SYNCHRONIZATION TECHNIQUES FOR DIGITAL DEMODULATION
SIGNAL MODEL

For a given modulation, the signal model is ‘a priori’ known:

\[ r(t) = s(t; \phi) + n(t) \quad \text{where } n(t) \text{ is an AWGN term} \]

and \[ \phi = (\tau_o, \theta_o, f_o, A_o, \{c_n\}) \]

\( \tau_o \) is the 'timing' error
\( \theta_o \) is the 'carrier phase' error
\( f_o \) is the 'carrier frequency' error
\( A_o \) is the 'signal amplitude'
\( \{c_n\} \) is the 'information symbol sequence'

Further on, we will assume a modulated digital PAM signal:

\[ s(t; \phi) = A_o \left[ \sum_n c_n h(t - nT - \tau_o) \right] e^{j\theta_o(t)} \quad \text{where } \theta_o(t) = \theta_o + 2\pi f_o t \]

where all the conventional synchronization parameters are reflected:
MCRB FOR SYNCHRONIZATION

\[ MCRB(\lambda) = \frac{1}{E_{r,u}} \left\{ \left( \frac{\partial \ln f(r \mid u, \lambda)}{\partial \lambda} \right)^2 \right\} \]

\[ f(r \mid u, \lambda) = \exp \left[ -\frac{1}{2N_0} \int_{T_o} \left| r(t) - s(t) \right|^2 dt \right] \]

\[ MCRB(\lambda) = \frac{N_0}{E_u} \left\{ \int_{T_o} \left| \frac{\partial s(t)}{\partial \lambda} \right|^2 dt \right\} \]

MCRB FOR FREQUENCY ESTIMATION

\[ \int_{T_0} \left| \frac{\partial s(t)}{\partial f_0} \right|^2 dt = 4\pi^2 \int_{T_0} t^2 \left| \sum_n c_n h(t - nT - \tau_o) \right|^2 dt \]

\[ E_{c,\tau_o} \left\{ \left| \sum_n c_n h(t - nT - \tau_o) \right|^2 \right\} = E_s \]

\[ E_{u,f} \left\{ \int_{T_0} \left| \frac{\partial s(t)}{\partial f_0} \right|^2 dt \right\} = \frac{2\pi^2 E_s T_0^3}{3T} \]

\[ \text{MCRB}(f_0) = \frac{3T}{2\pi^2 T_0^3} \frac{1}{E_s} / N_0 \]
MCRB FOR PHASE ESTIMATION

\[ MCRB(\theta_0) = \frac{N_0}{E_{u_\theta} \left\{ \int_{T_0} \left| \frac{\partial s(t)}{\partial \theta} \right|^2 dt \right\}} \]

\[ \int_{T_0} \left| \frac{\partial s(t)}{\partial \theta} \right|^2 dt = \int_{T_0} \sum_n c_n h(t - nT - \tau_o) \left| \frac{\partial s(t)}{\partial \theta_0} \right|^2 dt \]

\[ E_{u_\theta} \left\{ \int_{T_0} \left| \frac{\partial s(t)}{\partial \theta} \right|^2 dt \right\} = 2E_s L \quad \quad L = \frac{T_0}{T} \]

\[ MCRB(\theta_0) = \frac{1}{2L} \frac{1}{E_s / N_0} \]
MCRB FOR TIMING ESTIMATION

\[
MCRB(\tau_0) = \frac{1}{8\pi^2 \xi L} \frac{1}{E_s / N_0}
\]

\[
\xi = T^2 \frac{\int_{-\infty}^{\infty} f^2 |H(f)|^2 \, df}{\int_{-\infty}^{\infty} |H(f)|^2 \, df}
\]

\(\xi\) is the normalized mean square bandwidth of \(H(f)\)

Timing estimation should be easier with wideband signals.
MAXIMUM LIKELIHOOD ESTIMATION (MLE)

Joint ML estimation/detection rule: \( (\hat{c}, \hat{\theta}) = \arg \max_{c,\theta} f(r \mid c, \theta) \)

- Data-aided (DA): using a preamble of known symbols.
  \( \hat{\theta}(c_0) = \arg \max_{\theta} f(r \mid c = c_0, \theta) \)

- Decision-Directed (DD): using segments of decoded data as if it where the true symbols.
  \( \hat{\theta}(c_L) = \arg \max_{\theta} f(r \mid c = \hat{c}_L, \theta) \)

- Non-Data-Aided (NDA): Performing the averaging operation to remove the data dependency.
  \( f(r \mid \theta) = \sum_{all \ possible \ sequences \ c} f(r \mid c, \theta) P(c) \)
  \( \hat{\theta} = \arg \max_{\theta} f(r \mid \theta) \)
MAXIMUM LIKELIHOOD ESTIMATION (II)

- Under the AWGN assumption, the ML optimal parameter estimation corresponds to the conventional Minimum Mean Square Error (M.M.S.E.), that is, the received data and the signal model fitting:

\[ \min \varepsilon^2 = \min_{\phi} \int_{T_o} |r(t) - s(t; \phi)|^2 \, dt \]

- Thus:

\[ \min \varepsilon^2 = \min_{\phi} \left[ \int_{T_o} |r(t)|^2 \, dt + \int_{T_o} |s(t; \phi)|^2 \, dt - 2 \Re \left\{ \int_{T_o} s^*(t; \phi) r(t) \, dt \right\} \right] \]

- And for ‘constant envelope signals’, it is necessary to perform the multi-dimensional search as follows:

\[ \min \varepsilon^2 \equiv \max_{\phi} \Re \left\{ \int_{T_o} s^*(t; \phi) r(t) \, dt \right\} \]
DA ESTIMATION

- As we saw, the optimal parameters estimation becomes from maximizing:

\[
\min \varepsilon^2 \equiv \max_{\phi} \mathbb{R}\left\{ \int_{T_o} s^*(t; \phi) r(t) dt \right\}
\]

- Thus, we have to maximize:

\[
\max_{\theta_o} \mathbb{R}\left\{ \sum_{n} c_n^* e^{-j\hat{\theta}_o} \int_{T_o} r(t) h^*(t - nT - \tau_o) dt \right\} = \max_{\theta_o} \mathbb{R}\left\{ e^{-j\hat{\theta}_o} \sum_{n} c_n^* x(n) \right\}
\]

\[
\hat{\theta}_o = \arg \left\{ \sum_{n} c_n^* x(n) \right\}
\]
DD ESTIMATION

\[ \max_{\theta_o} \mu^2 = \max_{\theta_o} \Re \left\{ \sum_n \hat{c}_n^* e^{-j\hat{\theta}_o} \int_{T_0} r(t) h^* (t-nT-\tau_o) \, dt \right\} \]

\[ \frac{\partial}{\partial \hat{\theta}_o} \mu^2 = 0 \quad \text{where} \quad u_\theta \equiv \frac{\partial}{\partial \hat{\theta}_o} \mu^2 = \frac{\partial}{\partial \hat{\theta}_o} \Re \left\{ \sum_n \hat{c}_n^* e^{-j\hat{\theta}_o} \int_{T_0} r(t) h^* (t-nT-\tau_o) \, dt \right\} \]

ML Estimation: \[ u_\theta \equiv \Im \left\{ \sum_n \hat{c}_n^* e^{-j\hat{\theta}_o} \int_{T_0} r(t) h^* (t-nT-\tau_o) \, dt \right\} \]

Loop equations: \[ \hat{\theta}(n+1) = \hat{\theta}(n) + k_\theta u_\theta(n) \]
CLOSED LOOP SCHEME
**DD Schemes: BPSK/QPSK**

Sampled Matched Filter output:

\[ I_k + jQ_k = \int_{t_0}^{t} r(t)h^*(t-nT)dt \]

\[ u_\theta = \frac{\partial}{\partial \hat{\theta}_o} \mu^2 = \text{Im}\{c_n^* e^{-j\hat{\theta}_o} (I_k + jQ_k)\} \]

\[ (I'_k + jQ'_k) = e^{-j\hat{\theta}_o} (I_k + jQ_k) \]

\[ \hat{c}_n = \begin{cases} \text{BPSK: } & \text{sign}[I_k] \\ \text{QPSK: } & \text{sign}[I'_k] + jsign[Q'_k] \end{cases} \]

**BPSK**

\[ u_\theta = \text{Im}\{\text{sign}[I'_k] (I'_k + jQ'_k)\} \]

\[ u_\theta = \text{sign}[I'_k] Q'_k \]

**QPSK**

\[ u_\theta = \text{Im}\{[\text{sign}[I'_k] - jsign[Q'_k]](I'_k + jQ'_k)\} \]

\[ u_\theta = \text{sign}[I'_k] Q'_k - \text{sign}[Q'_k] I'_k \]
NDA Schemes: BPSK/QPSK

Sampled Matched Filter output:

\[ I_k + jQ_k = \int_{T_o}^{T_o} r(t) h^*(t-nT) dt \bigg|_{t=kT} \]

Use of a Non-Linearity

\[ c = e^{j2\pi l/M} \rightarrow c^M = 1 \]

\[
\text{max Re} \left\{ \sum_n c_n^* e^{-j\theta_o} \int_{T_o} r(t) h^* (t-nT+n_0) dt \right\} = \text{max Re} \left\{ e^{-j\theta_o} \sum_n c_n^* x(n) \right\}
\]

\[
\text{max Re} \left\{ e^{-j\theta_oM} \sum_n x(n)^M \right\} \rightarrow e^{-j\theta_oM} = \exp \left[j \arg \sum_n x(n)^M \right]
\]

\[ u_{\theta} = \frac{\partial}{\partial \hat{\theta}_o} \mu^2 = \text{Im} \left\{ \hat{c}_n e^{-j\hat{\theta}_o} (I_k + jQ_k) \right\} \]

\[ (I'_k + jQ'_k) = e^{-j\hat{\theta}_o} (I_k + jQ_k) \]

\[ (I''_k + jQ''_k) = [I'_k + jQ'_k]^M \quad \text{(for MPSK)} \]
Ambiguity: if \( \hat{\theta} \) is a solution so is \( \hat{\theta} + \frac{2\pi l}{M} \) for \( l = 2, \ldots, M-1 \)

\[
e^{j\left(\hat{\theta} + \frac{2\pi l}{M}\right)M} = e^{j\hat{\theta}M}
\]

**BPSK**

\[
u_{\theta} = \text{Im}\{I_k'' + jQ_k''\}
\]

- Phase ambiguity: \( \pi \)
- Frequency error: \( 2f_e \)

**QPSK**

\[
u_{\theta} = \text{Im}\{I_k'' + jQ_k''\}
\]

- Phase ambiguity: \( \frac{\pi}{2} \)
- Frequency error: \( 4f_e \)
CARRIER ACQUISITION AND TRACKING

\[ \left( I(k) + jQ(k) \right) e^{j\theta(k)} \]

\[ \left( I(k) + jQ(k) \right) e^{j(\theta(k) - \hat{\theta}(k))} \]

**NCO**

**MATCHED FILTER**

**SYMBOL DETECTION**

**TO DECISION (DECODING)**

**LOOP FILTER**

**CARRIER PHASE DISCRIMINATOR**
SIGNAL MODEL

- Let’s consider an I&Q pure sinusoid:

\[ r(t) = A_0 e^{j(2\pi f_c t + \phi_0 + \theta_0)} + n(t) \]

- The In-phase and the Quadrature components of the sampled signal becomes:

\[
I(k) = \frac{A_0}{2} \cos(\theta(k) - \hat{\theta}(k)) + \frac{1}{2} n_i(t) \]

\[
Q(k) = \frac{A_0}{2} \sin(\theta(k) - \hat{\theta}(k)) + \frac{1}{2} n_q(t) \]

- The phase error is defined by:

\[ \psi(k) = \theta(k) - \hat{\theta}(k) \]

- A typical carrier phase discriminator is (among others)

\[
e(k) = g[\psi(k)] = \arctan\left(\frac{Q(k)}{I(k)}\right) \quad |e(k)| \leq \pi
\]
CARRIER DISCRIMINATORS (I)

- A typical *carrier phase discriminator* is:

\[
e(k) = g[\psi(k)] = \arctan\left( \frac{Q(k)}{I(k)} \right) \quad |e(k)| \leq \pi
\]

or:

\[
e(k) = g[\psi(k)] = \text{imag}[I(k) + jQ(k)] = Q(k)
\]
CARRIER DISCRIMINATORS (II)

PHASE DISCRIMINATOR

\[ \theta(k) \rightarrow \psi(k) \rightarrow g[\psi(k)] \rightarrow e(k) \rightarrow F(z) \rightarrow y(k) \]

\[ \hat{\theta}(k) \rightarrow D(z) \rightarrow \Theta(z) = D(z)Y(z) \text{ with } D(z) = \frac{z^{-1}}{1-z^{-1}} \text{ (ideal integrator)} \]

\[ Y(z) = F(z)E(z) \]

\[ e(k) = g[\psi(k)] = \psi(k) = \theta(k) - \hat{\theta}(k) \Rightarrow |e(k)| = |\psi(k)| \leq \pi \]

\[ e(k) = g[\psi(k)] + n_\theta(k) \Rightarrow |e(k)| \leq \pi \]
FIRST ORDER LOOP FILTER (NOISE FREE)

\[
\begin{align*}
\Psi(z) &= \frac{1}{1 + F(z)D(z)} \Theta(z) \\
D(z) &= \frac{z^{-1}}{1 - z^{-1}} \quad \text{(NCO)}
\end{align*}
\]

\[
\psi(k) = (1 - G_1)\psi(k-1) + [\theta(k) - \theta(k-1)]
\]

\[
\Theta(z) = \frac{1}{1 + F(z)D(z)}
\]

\[
F(z) = G_1
\]
FIRST ORDER: TRACKING ERROR

\[ \psi(k) = (1 - G_1)\psi(k-1) + [\theta(k) - \theta(k-1)] \]

Filtrado/integración  Error/innovación

- We will consider a linear phase error evolution (carrier phase and frequency errors):

\[ \theta(k) = a_1 k + a_0 \rightarrow \text{Phase error} \]

Frequency error

- The basic difference equation becomes:

\[ \psi(k) = (1 - G_1)\psi(k-1) + a_1 \]

which defines the system transient response for a given initial  \( \psi(0) = \psi_0 \)

- Under steady-state conditions:

\[ \psi(k) = \psi(k-1) = \psi_{ss} \Rightarrow \psi_{ss} = (1 - G_1)\psi_{ss} + a_1 \Rightarrow \psi_{ss} = \frac{a_1}{G_1} \]

TRACKING ERROR
FIRST ORDER: LOCK-IN

\[ \psi(k) = (1 - G_1)\psi(k-1) + a_1 \]

- To ensure the system Lock-In, it is necessary to avoid any phase aliasing (wrapping) in the loop:

\[ |\psi(k)| \leq \pi \quad \forall k \]

- The worst cases are the following:

\( a_i > 0 \) (positive frequency error) with \( \psi(k-1) = \pm \pi \)

\[ \psi(k) = a_i + (1 - G_1)\pi < \pi \quad \Rightarrow \quad G_1 > \frac{a_i}{\pi} \]

\[ \psi(k) = a_i - (1 - G_1)\pi > -\pi \quad \Rightarrow \quad G_1 < 2 - \frac{a_i}{\pi} \]

\( a_i < 0 \) (negative frequency error) with \( \psi(k-1) = \pm \pi \)

\[ G_1 > \frac{-a_i}{\pi} \]

\[ G_1 < 2 + \frac{a_i}{\pi} \]

LOCK-IN: \[ \frac{|a_1|}{\pi} < G_1 < 2 - \frac{|a_1|}{\pi} \Rightarrow |a_i| < \pi \quad (Nyquist) \quad |1 - G_1| < 1 \]
FIRST ORDER LOOP FILTER (WITH NOISE)

PHASE DISCRIMINATOR

\[ \theta(k) \]
\[ \psi(k) \]
\[ \hat{\theta}(k) \]
\[ g[\psi(k)] \]
\[ n_\phi(k) \]
\[ e(k) \]
\[ F(z) \]
\[ y(k) \]

D(z)

NCO

\[ \psi(k) = \theta(k) - \hat{\theta}(k) \]
\[ e(k) = g[\psi(k)] + n_\phi(k) \]
\[ y(k) = G_i e(k) \]
\[ \hat{\theta}(k) = \hat{\theta}(k-1) + y(k-1) \]

FIRST ORDER MARKOV PROCESS

\[ \psi(k) = \psi(k-1) - G_i g[\psi(k-1)] + a_i - G_i n_\phi(k-1) \]
PRAGMATIC APPROACH

- If the discriminator disturbance noise is white: \( S_{n_\theta n_\theta}(f) = \frac{\sigma^2_{n_\theta}}{f_s} \)

\[
\Psi(z) = \frac{D(z)F(z)}{1 + D(z)F(z)} N_\theta(z)
\]

\[
\begin{align*}
E[\psi_{ss}] &\approx \frac{a_1}{G_1} \\
E[\psi^2_{ss}] &\approx \frac{T_s}{2\pi} \int_{-\pi}^{\pi} \left| \frac{D(z)F(z)}{1 + D(z)F(z)} \right|^2 \sigma^2_{n_\theta} \, df \\
\text{with} \quad T_s &= \frac{1}{f_s}
\end{align*}
\]

\[
\text{var}[\psi_{ss}] = E[\psi^2_{ss}] - E^2[\psi_{ss}]
\]

\[
\text{var}[\psi_{ss}] \approx \frac{G_1}{2 - G_1} \sigma^2_{n_\theta}
\]
PHASE DISTURBANCE MEAN POWER

\[ n_\theta(k) = \arctan \left[ \frac{n_q(k)}{A_0 + n_i(k)} \right] \text{ for } k = 1, 2, ..., N \]

\[ \hat{\sigma}_n^2 = \frac{1}{N} \sum_{k=1}^{N} |n_\theta(k)|^2 \]

where the noise complex samples are zero mean and gaussian

The discriminator noise disturbance has to be generated at the working SNR, that is:

\[ SNR = \frac{S}{N} = \frac{1}{E[n_i^2] + E[n_q^2]} = \frac{1}{2E[n_i^2]} = \frac{1}{2E[n_q^2]} \]
SECOND ORDER LOOP FILTER (NOISE FREE)

LOOPFILTER: 
\[ F(z) = G_1 + \frac{G_2}{1-z^{-1}} \]
\[ r = G_1 + \frac{G_2}{G_1} \]

\[ \Psi(z) = \frac{1}{1 + F(z)D(z)} \Theta(z) \]
\[ D(z) = \frac{z^{-1}}{1-z^{-1}} \text{ (NCO)} \]

\[ \psi(k) = 2\psi(k-1) - \psi(k-2) + \]
\[ + [\theta(k) - 2\theta(k-1) + \theta(k-2)] - \]
\[ - (G_1 + G_2)\psi(k-1) + G_1\psi(k-2) \]

- We will consider a \textit{quadratic} phase error evolution:

\[ \theta(k) = a_2k^2 + a_1k + a_0 \]

Phase error

Frequency Rate error

Frequency error

TRACKING ERROR: \[ \psi_{ss} = \frac{2a_2}{G_2} \quad \forall a_1 \text{!!!} \]

LOCK-IN:
\[ \begin{cases} 
2 + \left| \frac{2a_2}{\pi} \right| < G_1(r + 1) < 4 - \left| \frac{2a_2}{\pi} \right| \\
\left| \frac{2a_2}{\pi} \right| < G_1(r - 1) < 2 - \left| \frac{2a_2}{\pi} \right| \\
|a_2| < \frac{\pi}{2} \\
r > 1 
\end{cases} \]
SECOND ORDER LOOP FILTER (WITH NOISE)

\[ E[\psi_{ss}] \approx \frac{2a_2}{G_2} \]

\[ E[\psi_{ss}^2] \approx \frac{T_s}{2\pi} \int_{-\pi}^{\pi} \left| \frac{D(z)F(z)}{1 + D(z)F(z)} \right|^2 \frac{\sigma_{n\theta}^2}{f_s} df \quad \text{with} \quad T_s = \frac{1}{f_s} \]

\[ \text{var}[\psi_{ss}] = E[\psi_{ss}^2] - E^2[\psi_{ss}] \]

\[ \text{var}[\psi_{ss}] \approx \frac{2(r-1) + G_1(r+1)}{4 - G_1(r+1)} \sigma_{n\theta}^2 \]

NORMALIZED NOISE VARIANCE for \( r = [1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9 2.0] \)

NORMALIZED NOISE VARIANCE for \( r = [1.0 1.01 1.02 1.03 1.04 1.05 1.06 1.07 1.08 1.09 1.1] \)