Continuum and Fluid Mechanics

CHAPTER 5:
Basic equations of Fluid Mechanics

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OUTLINE

1. Solids and fluids
2. Fluid Statics
3. Perfect gas
4. Newtonian fluids
5. Navier-Stokes equations
6. Rotating frames
7. Bernoulli equation
1. Solids and fluids

**Solid:**
- has a preferred shape
- it takes another (constant) shape under the action of (constant) external forces
- it relaxes to that shape when the external forces are removed

**Fluid:**
- has not any preferred shape
- it changes shape continuously under the action of fixed external forces

This is so for shear stresses, but:
- for normal stresses fluids and solids behave similarly
- normal stresses can only be a compression (usually)
Fluids: \[
\begin{align*}
\textbf{Gases:} & \quad \text{always tend to expand and occupy} \\
& \quad \text{the entire volume of any container} \\
\textbf{Liquids:} & \quad \text{the volume does not change very much}
\end{align*}
\]

The distinction between solids and fluids apply well to many materials under normal conditions but:

- solids under very strong shear stresses may behave partially as fluids (plasticity)
- some fluids (like jelly, paint, polymer solutions, …) may behave partially as solids and have a partial memory of a ‘preferred shape’ (viscoelasticity)
2. Fluid statics (or Hydrostatics)

Fluid at rest:

It is empirically found that stresses are isotropic:

\[ \tau = -p \mathbf{1} \]

where \( p \) is > 0 (compression) and is called the pressure.

\[ \rho \frac{d\vec{v}}{dt} = \nabla \cdot \tau + \rho \vec{g} \]

momentum equation

In case of gravity:

\[ \begin{align*}
\frac{\partial p}{\partial x} &= 0, \\
\frac{\partial p}{\partial y} &= 0, \\
\frac{\partial p}{\partial z} &= -\rho g
\end{align*} \]

\[ p = -\rho g z + p_0 \]

if \( g = \text{const.} \), \( \rho = \text{const.} \)
3. Perfect gas

The density of fluids depends on pressure and temperature.

\[ \rho = \rho(p, T) \]

is called the equation of state

most gasses in normal conditions obey the following equation of state:

\[ p = \rho RT \]

\( T = \) absolute temperature
\( R = \frac{R_U}{m_m} \)
\( R_U = \) universal constant = 8.314 J mol\(^{-1}\) K\(^{-1}\)
\( m_m = \) molecular mass. For dry air = 28.97 Kg/kmol

Perfect gas = the limit case when this equation is verified exactly
Thermodynamic properties of perfect gases

Specific heat:

\[ C_v = \frac{1}{m} \left( \frac{dQ}{dT} \right)_{V=\text{const.}} \quad C_p = \frac{1}{m} \left( \frac{dQ}{dT} \right)_{p=\text{const.}} \]

\[ C_p - C_v = R_U \]

for air at ordinary temperatures:

\[ \gamma = 1.4 \quad \text{and} \quad C_p = 1005 \text{ J Kg}^{-1} \text{ K}^{-1} \]

For an adiabatic (no heat transfer) process:

\[ \frac{p}{\rho^\gamma} = \text{const.} \]

Thermal expansion coefficient

\[ \alpha = -\frac{1}{\rho} \left( \frac{\partial \rho}{\partial T} \right)_{p=\text{const.}} = \frac{1}{T} \]

The specific internal energy is a function of the temperature only:

\[ U = U(T) \]
4. Newtonian fluids

Problem of fluid motions

Equation of motion: Cauchy or momentum equations

\[
\rho \frac{d\vec{v}}{dt} = \nabla \cdot \vec{\tau} + \rho \vec{g}
\]

Usually, the body forces \( \vec{g} \) are known

- but, we need to know \( \tau(\vec{v}) \) in order to solve for the motion
Fluid at rest:
\[ \tau = -p \mathbf{1} \]

Fluid in motion:
\[ \tau = -p \mathbf{1} + \sigma \]

\( p \) = called thermodynamic pressure

additional stresses which are proportional to the strain rate:
\[ \sigma_{ij} = c_{ijmn} D_{mn} \]

If the fluid is isotropic:
\[ c_{ijmn} = \lambda \delta_{ij} \delta_{mn} + \mu \delta_{im} \delta_{jn} + \gamma \delta_{in} \delta_{jm} \]

\( \sigma_{ij} = \text{symmetric} \Rightarrow \mu = \gamma \Rightarrow \sigma_{ij} = 2\mu D_{ij} + \lambda D_{kk} \delta_{ij} \]

\[ \tau_{ij} = -p \delta_{ij} + \lambda D_{kk} \delta_{ij} + 2\mu D_{ij} \]
But the pressure was already defined from the mean normal stress
\[ p = -\frac{1}{3} \tau_{ii} \]
this is called the **mean pressure** and it is different from the thermodynamic one

\[ \tau_{ij} = -p \delta_{ij} + \lambda D_{kk} \delta_{ij} + 2\mu D_{ij} \]
\[ \Rightarrow \quad p = p - \left( \lambda + \frac{2}{3} \mu \right) D_{ii} \]
\[ p - \bar{p} = \left( \lambda + \frac{2}{3} \mu \right) \nabla \cdot \vec{v} \]

For **incompressible fluid**: \( \nabla \cdot \vec{v} = 0 \) \[ \Rightarrow \quad p = \bar{p} \]
\[ \Rightarrow \quad \tau_{ij} = -p \delta_{ij} + 2\mu D_{ij} \]

In general, bulk viscosity: \( \kappa = \lambda + \frac{2}{3} \mu \) is found to be \( \approx 0 \)

**Newtonian fluid**: \( \kappa = \lambda + \frac{2}{3} \mu = 0 \)
\[ \tau_{ij} = -\left( p + \frac{2}{3} \mu \nabla \cdot \vec{v} \right) \delta_{ij} + 2\mu D_{ij} \]
Air and water obey very well the Newtonian fluid model

**Examples of non Newtonian fluids:**
- solutions containing polymer molecules
- blood
- water with clay

→ Stresses are nonlinear functions of strain rates
→ Stresses depend not only on instantaneous values of strain rate but also on its history → material with memory → viscoelastic
\[ \mu = \text{viscosity coefficient} \]

it depends on the thermodynamic properties \((T, \rho)\)

**Meaning:** shear experiment

\[
\begin{align*}
\tau_{ij} &= -\left( p + \frac{2}{3} \mu \nabla \cdot \vec{v} \right) \delta_{ij} + 2\mu D_{ij} \\
\tau_{21} &= 2\mu D_{21} \quad \Rightarrow \quad \frac{F}{A} = 2\mu \frac{u}{2b} \\
\mu &= \frac{F}{A} \frac{u}{b}
\end{align*}
\]
**Meaning:** the viscosity tends to smooth out the gradients in velocity

\[
v = u
\]

\[
v_x = \frac{u}{b} y
\]

\[
D = \begin{pmatrix}
0 & u/2b & 0 \\
u/2b & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
\tau_{21} = 2\mu D_{21} = \mu \frac{u}{b} > 0
\]

faster particles are slowed down

slower particles are accelerated
Viscous dissipation

\[ \tau : D = -p \nabla \cdot \vec{v} + \tau' : D' \]

- deformation work
- work associated to changes in volume
- shear work

\[ \tau_{ij} = - \left( p + \frac{2}{3} \mu \nabla \cdot \vec{v} \right) \delta_{ij} + 2\mu D_{ij} \]

\[ \tau'_{ij} = 2\mu D'_{ij} \]

Second law of thermodynamics
\[ \tau' : D' \geq 0 \]

- the viscosity is always positive
- the viscosity always dissipates mechanical energy

\[ \mu \geq 0 \]
\[ 2\mu D'_{ij} D'_{ij} \geq 0 \]

viscous dissipation work
5. Navier-Stokes equations

Problem of fluid motions

**Equation of motion**

\[ \rho \frac{dv_i}{dt} = \frac{\partial \tau_{ji}}{\partial x_j} + \rho g_i \]

**Newtonian fluid**

\[ \tau_{ij} = -\left( p + \frac{2}{3} \mu \nabla \cdot \vec{v} \right) \delta_{ij} + 2\mu D_{ij} \]

\[
\rho \frac{dv_i}{dt} = -\frac{\partial p}{\partial x_i} + \rho g_i + \frac{\partial}{\partial x_j} \left( \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - \frac{2\mu}{3} \frac{\partial v_k}{\partial x_k} \delta_{ij} \right)
\]

\[ \mu = \text{const.} \quad \rightarrow \quad \rho \frac{dv_i}{dt} = -\frac{\partial p}{\partial x_i} + \rho g_i + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} + \frac{\mu}{3} \frac{\partial}{\partial x_i} \left( \frac{\partial v_j}{\partial x_j} \right) \]

**Navier-Stokes equations**

\[ \rho \frac{d\vec{v}}{dt} = -\nabla p + \rho \vec{g} + \mu \nabla^2 \vec{v} + \frac{\mu}{3} \nabla (\nabla \cdot \vec{v}) \]
but the pressure and the density changes are unknown $\rightarrow \rho$ and $p$ are also variables

continuity equation and state equation are involved:

$$\frac{d\rho}{dt} + \rho \nabla \cdot \vec{v} = 0$$

$$\rho = \rho(p, T)$$

$\Rightarrow T$ is also a variable!

equation for the temperature is needed: first law of Thermodynamics

$$\rho \frac{du}{dt} = \tau : \mathbf{D} - \nabla \cdot \vec{q} + \rho r$$

$u = u(T, \rho)$
Therefore, the dynamical problem involves at least (in general):

6 variables: velocity, pressure, density, temperature:

\[ \mathbf{v}(\mathbf{x}, t), p(\mathbf{x}, t), \rho(\mathbf{x}, t), T(\mathbf{x}, t) \]

6 equations:

\[
\begin{aligned}
\rho \frac{d\mathbf{v}}{dt} &= -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{v} + \frac{\mu}{3} \nabla(\nabla \cdot \mathbf{v}) \\
\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} &= 0 \\
\rho &= \rho(p, T) \\
\rho \frac{du}{dt} &= \mathbf{\tau} : \mathbf{D} - \nabla \cdot \mathbf{q} + \rho r \\
\end{aligned}
\]

\[ u = u(T, \rho) \]
If viscosity can be neglected:

\[\rho \frac{d\vec{v}}{dt} = -\nabla p + \rho \ddot{g}\]

**Euler equations**

\[\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + g_i\]
6. Rotating frames

\[ \ddot{u} = u_i \dot{e}_i = u'_i \dot{e}'_i \quad \Rightarrow \quad \begin{pmatrix} \frac{d\dot{u}}{dt} \end{pmatrix}_S = \frac{du'_i}{dt} \dot{e}'_i + u'_i \left( \frac{d\dot{e}'_i}{dt} \right)_S = \left( \frac{d\ddot{u}}{dt} \right) + u'_i \left( \frac{d\dot{e}'_i}{dt} \right)_S \]

\[ \left( \frac{d\dot{e}'_i}{dt} \right)_S = ? \quad \Rightarrow \quad \dot{e}'_i = Q_{ji} \dot{e}_j \quad \Rightarrow \quad \frac{d\dot{e}'_i}{dt} = \frac{dQ_{ji}}{dt} \dot{e}_j = \frac{dQ_{ji}}{dt} Q_{jk} \dot{e}'_k = W_{ki} \dot{e}'_k \]

\[ \Omega_i = -\frac{1}{2} \varepsilon_{ijk} W_{jk} \]

\[ \ddot{\Omega} = \text{angular velocity of frame } S' \text{ relative to frame } S \]

\[ \begin{pmatrix} \dot{\Omega} \end{pmatrix}_S = \ddot{\Omega}^T = \dot{\Omega} \times \ddot{u} \]
\[ \ddot{x} = \ddot{O}O' + \ddot{x}' \]

\[ \frac{d\ddot{x}'}{dt} = \frac{d\ddot{x}'}{dt} + \ddot{\Omega} \times \ddot{x}' \]

\[ \ddot{v} = \ddot{v}_{O O'} + \ddot{v}' + \ddot{\Omega} \times \ddot{x}' \]

\[ \dddot{a} = \dddot{a}_{O O'} + \dddot{a}' + \ddot{\Omega} \times (\ddot{\Omega} \times \ddot{x}) + 2 \dddot{\Omega} \times \dddot{v}' + \frac{d\dddot{\Omega}}{dt} \times \dddot{x}' \]

- centripetal acceleration
- Coriolis acceleration
Navier Stokes equations in the rotating frame

\[ \rho \frac{d\vec{v}'}{dt} = -\nabla p + \rho \ddot{\vec{g}}' + \mu \nabla^2 \vec{v} + \frac{\mu}{3} \nabla(\nabla \cdot \vec{v}) - 2\rho \vec{\Omega} \times \vec{v}' \]

with

\[ \ddot{\vec{g}}' = \ddot{\vec{g}} - \vec{\Omega} \times (\vec{\Omega} \times \vec{x}') - \vec{a}_{O'O} - \frac{d\vec{\Omega}}{dt} \times \vec{x}' \]

Coriolis force

centrifugal force / \( \rho \)
7. Bernouilli equation

Simplified expression of the Euler equations
(equations of motion for inviscid fluid, i.e., viscosity is neglected)

Assumptions:

- no viscosity, \( \mu = 0 \)
- barotropic flow, i.e., \( \rho = \rho(p) \)
- body force is gravity (= const.)

Euler equations:

\[
\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho} \nabla p + \vec{g}
\]

- \( \vec{g} = -\nabla (gx_3) \)
- \( \vec{v} \cdot \nabla \vec{v} = \vec{\omega} \times \vec{v} + \nabla \left( \frac{1}{2} \vec{v}^2 \right) \) vector identity already proven
- \( F(p) \equiv \int \frac{dp}{\rho(p)} \Rightarrow \nabla F = \frac{dF}{dp} \nabla p = \frac{1}{\rho} \nabla p \)
\[ \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho} \nabla p + \vec{g} \]

- \( \vec{g} = -\nabla (g x_3) = -\nabla (g z) \)
- \( \vec{v} \cdot \nabla \vec{v} = \vec{\omega} \times \vec{v} + \nabla \left( \frac{1}{2} v^2 \right) \)
- \( F(p) = \int \frac{dp}{\rho(p)} \Rightarrow \nabla F = \frac{dF}{dp} \nabla p = \frac{1}{\rho} \nabla p \)

\[ \frac{\partial \vec{v}}{\partial t} + \nabla \Theta = \vec{v} \times \vec{\omega} \]  

with \( \Theta = \frac{1}{2} v^2 + g z + \int \frac{dp}{\rho(p)} \)

**Bernoulli function**

This equation is specially useful in two particular cases:
1. Steady flow
2. Irrotational flow
Bernoulli equation for steady flow

\[ \frac{\partial \vec{v}}{\partial t} + \nabla \Theta = \vec{v} \times \vec{\omega} \]

\[ \nabla \Theta = \vec{v} \times \vec{\omega} \quad \Rightarrow \quad \nabla \Theta \perp \vec{v} \]

\[ \Theta = \frac{1}{2} v^2 + g z + \int \frac{dp}{\rho(p)} = \text{constant along the streamlines} \]

The constant may be different for different streamlines

\[ \vec{\omega} = 0 \quad \Rightarrow \quad \text{the constant is the same everywhere in the flow} \]
Bernoulli equation for irrotational flow

Irrotational flow: \( \vec{\omega} = \nabla \times \vec{v} = 0 \) \iff \( \exists \phi \mid \vec{v} = \nabla \phi \)

Potential flow

\[
\frac{\partial \vec{v}}{\partial t} + \nabla \Theta = \vec{v} \times \vec{\omega}
\]

\[
\frac{\partial}{\partial t} (\nabla \phi) + \nabla \Theta = 0
\]

\[
\nabla \left( \frac{\partial \phi}{\partial t} + \frac{1}{2} v^2 + g z + \int \frac{dp}{\rho (p)} \right) = 0
\]

\[
\frac{\partial \phi}{\partial t} + \frac{1}{2} v^2 + g z + \int \frac{dp}{\rho (p)} = F(t)
\]
Application of Bernouilli: Pitot tube

device to measure a flow velocity

- steady flow
- viscosity is neglected
- \( \rho = \text{const.} \)

\[
\frac{1}{2}v_1^2 + gz_1 + \frac{p_1}{\rho} = \frac{1}{2}v_2^2 + gz_2 + \frac{p_2}{\rho}
\]

inside the vertical tubes there is hydrostatic balance:

\( p_1 = p_{atm} + \rho gh_1 \), \( p_2 = p_{atm} + \rho gh_2 \)

\[
v_1 = \sqrt{2g(h_2 - h_1)}
\]
Application of Bernouilli: orifice in a tank

- orifice small ⇒ flow approximately steady
- viscosity is neglected
- \( \rho = \text{const.} \)

\[
\frac{1}{2} v_1^2 + gz_1 + \frac{p_1}{\rho} = \frac{1}{2} v_2^2 + gz_2 + \frac{p_2}{\rho}
\]

\[
gh = \frac{1}{2} v_2^2 \quad \Rightarrow \quad v_2 = \sqrt{2gh}
\]

\[
\frac{dm}{dt} = \rho A \sqrt{2gh}
\]

Actually, the function

\[
\Theta = \frac{1}{2} v^2 + gz + \frac{p}{\rho}
\]

has the same value everywhere since it has the same value at any point on the free surface ⇒ flow is irrotational

\[
\frac{\partial \vec{v}}{\partial t} + \nabla \Theta = \vec{v} \times \vec{\omega}
\]

\[
\vec{v} \times \vec{\omega} = 0 \quad \Rightarrow \quad \vec{\omega} = 0
\]