Exercise (except from an exam)
We consider the vector space \((\mathbb{R}^2, +, \cdot)\). Each vector \((x, y) \neq (0, 0)\)
is written in polar coordinates as
\[
\begin{align*}
x &= r \cos(\theta) \\
y &= r \sin(\theta)
\end{align*}
\]
where \(r > 0\) and \(\theta \in [0, 2\pi)\). Consider
the set \(E = \{(0, 0)\} \cup \{(r \cos(\theta), r \sin(\theta)) \in \mathbb{R}^2 / r > 0 \)
and \(\theta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right] \cup \left[\frac{5\pi}{4}, \frac{7\pi}{4}\right]\}\)

Is \(E\) a subspace of \(\mathbb{R}^2\)?
$u_1 = (1, 1) \in \mathbb{E}$
(angle $\theta = \frac{\pi}{4}$)

$u_2 = (1, -1) \in \mathbb{E}$
(angle $\pi + \frac{3\pi}{4}$)

$u_1 + u_2 = (1, 1) + (1, -1) = (2, 0)$

Thus we have found $u_1 \in \mathbb{E}$, $u_2 \in \mathbb{E}$ such that $u_1 + u_2 \in \mathbb{E}$

This means that $\mathbb{E}$ is NOT a subspace of $\mathbb{R}^2$

Another method:
We know that the only subspaces of $\mathbb{R}^2$ are: $\{(0,0)\}$,
the lines that go through (0,0) and $\mathbb{R}^2$. Since $E$ is none of these it cannot be a subspace of $\mathbb{R}^2$.

Student's answer

If we want $E$ to be a subspace of $\mathbb{R}^2$, it has to follow this:

\[
\begin{align*}
\forall x, y \in E & \implies (x + y) \in E \quad \text{(1)} \\
\forall x \in E, \forall \lambda \in \mathbb{R} & \implies \lambda x \in E \quad \text{(2)}
\end{align*}
\]

Incorrect

\[
\begin{align*}
\forall x, y \in E \quad & \text{we have } x + y \in E \\
\forall x \in E, \forall \lambda \in \mathbb{R} \quad & \text{we have } \lambda x \in E
\end{align*}
\]
We will consider

\[ \vec{V} = (r_x \cos \theta, r_x \sin \theta) \]

\[ \vec{u} = (r_y \sin \theta, r_y \cos \theta) \]

\[ \uparrow \quad \uparrow \quad \uparrow \]

? 

Condition 1

\[ \vec{V} + \vec{u} = (r_x \cos \theta + r_y \sin \theta, r_x \sin \theta + r_y \cos \theta) \]

Correct

\[ \forall x, y \in \mathbb{R} \]

\[ \uparrow \quad \uparrow \quad \uparrow \]

arbitrary

\[ \cos \left( \theta \right) \]

\[ \cos \left( \theta + r \sin \theta \right) \]
\[ \vec{v} = (r \cos \theta, r \sin \theta) \]

\[ \vec{u} = (r \sin \theta, r \cos \theta) \]

We cannot prove that \( \vec{u} \) is in \( E \).

\[ \text{Condition 2} \]

\[ \alpha \vec{v} = (\alpha r \cos \theta, \alpha r \sin \theta) \]

**Correct:** Let's consider some \( \vec{v} \in \mathbb{R}^2 \) and write \( \vec{v} \) in polar coordinates as

\[ \vec{v} = (r \cos \theta, r \sin \theta) \]

We can prove that it is not a subspace with numbers by showing that Condition 2 is not satisfied.
Let's take angle $\frac{\pi}{2} = \theta$
and $\alpha = -3$

$-3v = (-3r \cos \frac{\pi}{2}, -3r \sin \frac{\pi}{2})$

$\alpha \in \mathbb{R} \Rightarrow \text{so it can be negative}$

$r > 0$

So $\alpha \cdot v \notin \mathbb{R}$.

If $< 0 \Rightarrow$ there is a problem.

$$-3r \sin \frac{\pi}{2} = 3r \sin \left(\frac{\pi + \pi}{2}\right)$$
The argument of the student that \( x \notin E \) **IS IN CORRECT** conclusion of the student: 

\( E \) is not a subspace.