Control & Guidance

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Classical control
Classical control

1. Parametric estimation
2. Steady state error
3. Root locus
4. Controllers
5. Frequency response
6. Bode diagrams
Study properties of the response of the system:

- desired angle of attack $\alpha_{\text{ref}}$
- speed $u_{\text{ref}}$
- displacement of elevator $\delta_e$
- actual angle of attack $\alpha$
- speed $u$

1- Parametric estimation
1- Parametric estimation

Temporal methods:

a. First-order systems

b. Second-order systems

c. Higher-order systems
1- Parametric estimation

a. First-order systems

A first-order system is defined by a first-order differential equation:

\[ \tau \dot{y}(t) + y(t) = Kr(t) \]

\[ \rightarrow \quad G(s) = \frac{Y(s)}{R(s)} = \frac{K}{1 + \tau s} \]

\( \tau \): system time constant

\( K \): gain

Electrical/mechanical examples
1- Parametric estimation

a. First-order systems

Impulse response

\[ L[\delta(t)] = R(s) = 1 \rightarrow Y(s) = \frac{K}{1 + \tau s} \]

using the inverse Laplace transform, the impulse response is:

\[ y(t) = \frac{K}{\tau} e^{-t/\tau} \quad t \geq 0 \]
1- Parametric estimation

a. First-order systems

**Impulse response**

\[ y(t) = \frac{K}{\tau} e^{-t/\tau}, \quad t \geq 0 \]

Tangent slope in 0:

\[ \left. \frac{dy(t)}{dt} \right|_{t=0} = -\frac{K}{\tau^2} \]
1- Parametric estimation

a. First-order systems

**Step response**: response to a unit step function

\[
L[u(t)] = R(s) = \frac{1}{s} \quad \rightarrow \quad Y(s) = \frac{K}{(1 + \tau s)s} = \frac{K}{s} - \frac{K\tau}{1 + \tau s}
\]

using the inverse Laplace transform, the **step response** or **indicial response** is:

\[
y(t) = K \left( 1 - e^{-\frac{t}{\tau}} \right) \quad t \geq 0
\]
1- Parametric estimation

a. First-order systems

Step response:

\[ y(t) = K \left(1 - e^{-t/\tau}\right) \]

Tangent slope in 0:

\[ \frac{dy(t)}{dt} \bigg|_{t=0} = \frac{K}{\tau} \]
b. Second-order systems

A second-order system is defined by a second-order differential equation:

\[ b_2 \ddot{y}(t) + b_1 \dot{y}(t) + b_0 y(t) = a_0 r(t) \]

\[
G(s) = \frac{Y(s)}{R(s)} = \frac{a_0}{b_2 s^2 + b_1 s + b_0}
\]
b. Second-order systems

It can be factorized to emphasize particular parameters:

\[
G(s) = \frac{Y(s)}{R(s)} = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\]

with

- \(K\): system gain (corresponds to final value for a unit step function)
- \(\omega_n\): undamped natural frequency
- \(\zeta\): damping factor (\(\zeta > 0\))
1- Parametric estimation

b. Second-order systems

Step response:

\[
\frac{Y(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\]

Response depends on the poles of the transfer function

\[
s^2 + 2\zeta\omega_n s + \omega_n^2 = 0
\]

\[
s = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2 \omega_n^2 - 4\omega_n^2}}{2}
\]

let \( \Delta = 4\zeta^2 \omega_n^2 - 4\omega_n^2 = 4\omega_n^2(\zeta^2 - 1) \)

\[\rightarrow\] discriminant’s sign depends on \( \zeta \) value

\[\rightarrow\] poles and response's properties depend on \( \zeta \) value
b. Second-order systems

\( \zeta > 1 \) Over-damped movement (non-oscillatory modes)

Real and negative poles:

\[
S_{1,2} = \omega_n \left( -\zeta \pm \sqrt{\zeta^2 - 1} \right)
\]

\[
Y(s) = \frac{1}{s} \times \frac{K \omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} = \frac{1}{s} \times \frac{K \omega_n^2}{(s - s_1)(s - s_2)}
\]

Development in simple fractions:

\[
Y(s) = K \left( \frac{1}{s} + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \left( \frac{1}{s_1(s - s_1)} - \frac{1}{s_2(s - s_2)} \right) \right)
\]
b. Second-order systems

\( \zeta > 1 \) Over-damped movement (non-oscillatory modes)

Inverse Laplace transform:

\[
y(t) = K \left( 1 + \frac{\omega_n}{2 \sqrt{\zeta^2 - 1}} \left( \frac{e^{s_1 t}}{s_1} - \frac{e^{s_2 t}}{s_2} \right) \right)
\]

Tangent slope in 0:

\[
\left. \frac{dy(t)}{dt} \right|_{t=0} = 0
\]
b. Second-order systems

\[ \zeta = 1 \text{ Critically damped movement (non-oscillatory modes)} \]

Double real negative poles: 

\[ s_{1,2} = -\omega_n \]

\[
Y(s) = \frac{1}{s} \times \frac{K\omega_n^2}{(s + \omega_n)^2}
\]

Development in simple fractions:

\[
Y(s) = K \left( \frac{1}{s} - \frac{\omega_n}{(s + \omega_n)^2} - \frac{1}{s + \omega_n} \right)
\]
b. Second-order systems

$\zeta = 1$ Critically damped movement (non-oscillatory modes)

Inverse Laplace transform:

$$y(t) = K \left( 1 - (1 + \omega_n t) e^{-\omega_n t} \right)$$

Tangent slope in 0:

$$\left. \frac{dy(t)}{dt} \right|_{t=0} = 0$$
b. Second-order systems

\( \zeta < 1 \) Under-damped movement (oscillatory modes)

Conjugated complex poles:

\[
s_{1,2} = \omega_n \left( -\zeta \pm j\sqrt{1 - \zeta^2} \right)
\]

\[
Y(s) = \frac{1}{s} \times \frac{K\omega_n^2}{s \left[ (s + \zeta \omega_n)^2 + \omega_n^2 (1 - \zeta^2) \right]}
\]

Development in simple fractions…
b. Second-order systems

ξ<1 Under-damped movement: Inverse Laplace transform:

\[ y(t) = K \left( 1 - e^{-\zeta \omega_n t} \left( \cos(\omega_n \sqrt{1 - \zeta^2} t) + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin(\omega_n \sqrt{1 - \zeta^2} t) \right) \right) \]

Tangent slope in 0:

\[ \frac{dy(t)}{dt} \bigg|_{t=0} = 0 \]
1- Parametric estimation

b. Second-order systems

Characteristic parameters

\( K \): gain \[ \frac{\text{output final value}}{\text{input final value}} \]

\( M \): maximum overshoot: represents the value of the highest peak of the system response measured with respect to the reference value (final value)

\( t_p \): peak time: time needed for the response to arrive at its first peak

\( T \): period

\( t_s \): settling time
b. Second-order systems

Characteristic parameters: for second-order systems

\[
K = \text{e}^{-\frac{\pi \tan \varphi}{\tan \varphi}} = \text{e}^{-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}}
\]

\[
M = \text{e}^{-\frac{\pi}{\tan \varphi}} = \text{e}^{-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}}
\]

\[
t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}
\]

\[
T = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1-\zeta^2}}
\]

\[
\omega_d = \omega_n \sqrt{1-\zeta^2}
\]

(damped natural frequency)
1- Parametric estimation

b. Second-order systems

Obtain:
- $K$
- $M$
- $t_p$
- $t_{s5\%}$
- $T$

Deduce:
- $\zeta$
- $\omega_n$

Calculate:
- $G(s)=\frac{Y(s)}{R(s)}$
1- Parametric estimation

c. Higher-order systems

→ characterize the transitory state of any-order systems
generally $y(t)$ = linear combination of elementary time functions
defined by the nature (real or complex) of the characteristic equation roots: system modes:

• **real poles**: non-oscillatory modes, exponential term in the response

• **complex poles**: oscillatory modes, exponential term multiplied by sine or cosine
c. Higher-order systems

High order systems can be simplified using:

**dominant poles**

- Poles further from the imaginary axis have a weaker contribution.

**1 pole near 1 zero**

- If there is a zero near a pole, this pole contribution will be weak.
Study error of the response of the system:

- desired angle of attack $\alpha_{\text{ref}}$
- speed $u_{\text{ref}}$
- displacement of elevator $\delta_e$
- actual angle of attack $\alpha$
- speed $u$

2- Steady state error
2- Steady state error

Steady State error:

\[ e_{ss} = \text{difference between the entry signal and the exit signal} \]

\[ e_{ss} = \text{“what we want minus what we get”} \]
System’s type:

Given the transfer function:

$$G(s) = K \frac{(1 + as)(1 + bs)...(1 + cs + ds^2)...}{s^N (1 + \alpha s)(1 + \beta s)...(1 + \chi s + \delta s^2)...}$$

with \(K\): system gain,

and \(N\): number of poles in the origin

$$\rightarrow N = \text{system’s type}$$
Definition of Steady State error:

\[ e_{ss} = \lim_{t \to +\infty} [r(t) - y(t)] \]

\( e_{ss} > 0 \): exit signal has not reached the entry reference

\( e_{ss} < 0 \): exit signal is higher than the entry
2- Steady state error

\[ e_{ss} = \lim_{t \to +\infty} [r(t) - y(t)] \]

Moving to the Laplace space: **Final value theorem**:

\[ e_{ss} = \lim_{s \to 0} (s[R(s) - Y(s)]) = \lim_{s \to 0} \left( s \left[ R(s) - \frac{G(s)}{1 + G(s)} R(s) \right] \right) \]

\[ e_{ss} = \lim_{s \to 0} \left( s \times \frac{R(s)}{1 + G(s)} \right) \]

Depends on the entry + on the system’s type
2- Steady state error

\[ G(s) = K \frac{(1 + as)(1 + bs)...(1 + cs + ds^2)...}{s^N (1 + \alpha s)(1 + \beta s)...(1 + \chi s + \delta s^2)...} \]

1. Position error: error for a step function entry: \( r(t)=u(t) \)

\[ e_p = \lim_{s \to 0} \left( s \times \frac{1}{1 + G(s)} \times \frac{1}{s} \right) = \lim_{s \to 0} \left( \frac{1}{1 + G(s)} \right) = \frac{1}{1 + \lim_{s \to 0} G(s)} \]

\[ = \begin{cases} 
\frac{1}{1 + K} & \text{type 0} \\
0 & \text{type } \geq 1
\end{cases} \]
2- Steady state error

1. Position error: error for a step function entry: \( r(t) = u(t) \)
2- Steady state error

\[ G(s) = K \frac{(1 + as)(1 + bs)\cdots(1 + cs + ds^2)\cdots}{s^N(1 + \alpha s)(1 + \beta s)\cdots(1 + \chi s + \delta s^2)\cdots} \]

2. Speed error: error for a ramp function entry: \( r(t) = t \)

\[ e_v = \lim_{s \to 0} \left( s \times \frac{1}{1 + G(s)} \times \frac{1}{s^2} \right) = \lim_{s \to 0} \left( \frac{1}{s + sG(s)} \right) = \lim_{s \to 0} \left( \frac{1}{sG(s)} \right) \]

\[ = \begin{cases} 
\infty & \text{type 0} \\
\frac{1}{K} & \text{type I} \\
0 & \text{type } \geq \text{II}
\end{cases} \]
2- Steady state error

\[ G(s) = K \frac{(1 + as)(1 + bs) \ldots (1 + cs + ds^2) \ldots}{s^N (1 + \alpha s)(1 + \beta s) \ldots (1 + \chi s + \delta s^2) \ldots} \]

3. Acceleration error: error for a parabolic entry: \( r(t) = t^2 \)

\[ e_v = \lim_{s \to 0} \left( s \times \frac{1}{1 + G(s)} \times \frac{1}{s^3} \right) \]

\[ = \lim_{s \to 0} \left( \frac{1}{s^2 + s^2 G(s)} \right) \]

\[ = \lim_{s \to 0} \left( \frac{1}{s^2 G(s)} \right) \]

\[ = \begin{cases} 
\infty & \text{type 0 or I} \\
\frac{1}{K} & \text{type II} \\
0 & \text{type } \geq \text{III} 
\end{cases} \]
## Error based on type + entry

<table>
<thead>
<tr>
<th>Input: Type:</th>
<th>step</th>
<th>ramp</th>
<th>parabolic</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>constant</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>I</td>
<td>0</td>
<td>constant</td>
<td>$\infty$</td>
</tr>
<tr>
<td>II</td>
<td>0</td>
<td>0</td>
<td>constant</td>
</tr>
</tbody>
</table>
2- Steady state error

Example:

\[ \theta_{\text{ref}}(s) + \theta(s) = K \delta_e(s) G(s) \]

Compute the error in steady state for a unit step function entry and for a system with the following open loop transfer function:

\[ \frac{\theta(s)}{\delta_e(s)} = \frac{2s + 0.1}{s^2 + 0.1s + 4} \]

- for \( K=1 \), \( K=10 \), \( K=100 \),
- for \( K=1 \) and

\[ \frac{\theta(s)}{\delta_e(s)} = \frac{2s + 0.1}{s(s^2 + 0.1s + 4)} \]
Design a simple proportional controller in order to satisfy some constraints on the response of the system.

- desired angle of attack $\alpha_{\text{ref}}$
- speed $u_{\text{ref}}$
- displacement of elevator $\delta_e$
- actual angle of attack $\alpha$
- speed $u$

LONGITUDINAL CONTROL SYSTEM

LONGITUDINAL DYNAMICS

SENSORS: INS, Anemometer

3- Root locus
3- Root locus

• Root locus technique
• Gain setting
• Effect of zeros and poles
3- Root locus

Root locus technique

• Introduced by W. R. Evans in 1949: developed a series of rules that allow the control system engineer to quickly draw the root locus diagram = locus of all possible roots of the characteristic equation: \( 1+K G(s)= 0 \)

= locus of all possible poles in closed loop as K varies from 0 to infinity

• The resulting plot helps us in selecting the best value of K

• Gives information for the transitory part of the response (stability, damping factor, natural frequency)
3- Root locus

Root locus technique

Let

\[ G(s) = \frac{\alpha(s + z_1)(s + z_2)(s + z_m)}{(s + p_1)(s + p_2)(s + p_n)} \]

And substitute it in the characteristic equation

\[ 1 + \frac{k(s + z_1)(s + z_2)(s + z_m)}{(s + p_1)(s + p_2)(s + p_n)} = 0 \quad \text{where} \quad k = K\alpha \]
The characteristic equation is complex and can be written in terms of magnitude and angle as follows:

\[ \frac{|k(s + z_1)(s + z_2)\cdots(s + z_m)|}{|s + p_1|(s + p_2)\cdots|s + p_n|} = 1 \]

\[ \sum_{i=1}^{m} \angle(s + z_i) - \sum_{i=1}^{n} \angle(s + p_i) = (2q + 1) \times 180 \]

for \( q = 0, 1, 2 \ldots, (n - m - 1) \)
3- Root locus

Root locus technique: Rules

If we rearrange the magnitude criteria as

\[
\frac{|s + z_1||s + z_2||s + z_m|}{|s + p_1||s + p_2||s + p_n|} = \frac{1}{k}
\]

**Rule 1:** The number of separate branches of the root locus plot is equal to the number of poles of the transfer function (n)

Branches of the root locus **originate** at the poles of G(s) for k=0 and **terminate** at either the open-loop zeroes or at infinity for k=+∞

n separate branches, n-m infinite branches, m finite branches
3- Root locus

Root locus technique: Rules

Rule 2: Because the complex poles are always "conjugated",

the root locus branches are **symmetric** with respect to the real axis.
3- Root locus

Root locus technique: Rules

Rule 3:

Segments of the real axis that are part of the root locus:
points on the real axis that have an odd number of poles and zeroes to their right
3- Root locus

Root locus technique: Rules

Rule 4: Asymptotes

The root locus branches that approach the open-loop zeroes at infinity do so along straight-line asymptotes that intersect the real axis at the center of gravity of the finite poles and zeroes

\[
\sigma = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n - m}
\]

The angle that the asymptotes make with the real axis is given by

\[
\phi_a = \frac{180^\circ \left[2q + 1\right]}{n - m}
\]

for \(q = 0, 1, 2 \ldots, (n - m - 1)\)
3- Root locus

Root locus technique: Rules

Rule 5: breakaway points

If a portion of the real axis is part of the root locus and a branch is between two poles the branch must break away from the real axis so that the locus ends on a zero as $k$ approaches infinity. The breakaway points on the real axis are determined by solving

$$1 + \frac{k(s + z_1)(s + z_2)(s + z_m)}{(s + p_1)(s + p_2)(s + p_n)} = 0 \quad \text{for } k$$

and then finding the roots of the equation $\frac{dk}{ds}=0$

Only roots that lie on a branch of the locus are of interest.
Rule 6: Intersection with the imaginary axis

Solve the characteristic equation for \( s=j\omega \) (equation of the imaginary axis)

\[
1 + \frac{k(j\omega + z_1)(j\omega + z_2)(j\omega + z_m)}{(j\omega + p_1)(j\omega + p_2)(j\omega + p_n)} = 0
\]
3- Root locus

Root locus technique: Rules

Rule 7: for complex poles and zeroes only:

The angle of departure of the root locus from a pole of $G(s)$ or arrival angle at a zero of $G(s)$ can be found by the following expression

If you consider a test point $t$:

$$\sum_{i=1}^{m} \angle(t + z_i) - \sum_{i=1}^{n} \angle(t + p_i) = \pm 180$$
Example 1:

\[ G(s) = \frac{s + 2}{s^2 + 2s + 3} \]
Example 2:

\[ G(s) = \frac{1}{s(s+1)(s+2)} \]
3- Root locus

Setting of the gain and natural frequency

Basic operation: to adjust the gain $K$ to obtain a **damping factor**
given by the poles in closed-loop and fixed by the damping factor $\zeta$

$$\zeta = \frac{|\text{Re}(s_i)|}{|s_i|}$$

Cf second-order systems: straight line doing an angle $\varphi$ with the real axis ($\cos \varphi = \zeta$) sets an intersection point with the poles position, and $k$ (and then $K$) is obtained solving the characteristic equation

**Natural frequency** for a second-order system: $\omega_n = |s_i|$
3- Root locus

Gain setting

\[ 1 + KG(s) = 0 \]

\[ 1 + K \frac{\alpha(s + z_1)(s + z_2)(s + z_m)}{(s + p_1)(s + p_2)(s + p_n)} = 0 \]

The system total gain is computed thanks to the module condition

\[ k = K \times \alpha = \frac{p_1s \times p_2s \times p_ms}{z_1s \times z_2s \times z_ns} \]

“total gain” = product of the distances from the poles of \( G(s) \) to the intersection point (= target pole) divided by the product of the distances from zeros of \( G(s) \).
3- Root locus

Gain setting

\[ 1 + KG(s) = 0 \]

\[ 1 + K \frac{\alpha}{(s + p_1)(s + p_2)(s + p_n)} = 0 \]

If there are no zeros:

\[ K \times \alpha = p_1s \times p_2s \times p_m s \]

“total gain” = product of the distances between the poles of G(s) and the intersection point (= target pole)

Examples
Gain setting

\[ G(s) = \frac{1}{s(s + 1)(s + 2)} \]

Designer requirement: we want \( \zeta = 0.5 \)

\( \zeta = 0.5 = \cos \varphi \rightarrow \varphi = 60° \)
3- Root locus

Gain setting

\[ G(s) = \frac{1}{s(s + 1)(s + 2)} \]

Designer requirement:
we want \( \zeta = 0.5 \)

\[ \zeta = 0.5 = \cos \phi \rightarrow \phi = 60^\circ \]
Gain setting

$$G(s) = \frac{1}{s(s+1)(s+2)}$$

Designer requirement: we want $\zeta = 0.5$

This corresponds to $k = 1.04$

Examples
Gain setting

\[ G(s) = \frac{s + 2}{s^2 + 2s + 3} \]

Designer requirement: we want \( \zeta = 0.7 \)
Gain setting

\[ G(s) = \frac{s + 2}{s^2 + 2s + 3} \]

Designer requirement: we want \( \zeta = 0.7 \)

This corresponds to \( k = 1.32 \)

Examples
Relative stability: gain margin

“Re(s)<0” criterion informs about the absolute stability of a system but it says nothing about its relative stability

= how far it is from the instability → system strength

Gain margin: maximum proportional factor that can be introduced into the control loop until the system becomes critically stable.

\[ M_G = \frac{k_{Cr}}{k_{actual}} \]
3- Root locus

Gain setting

\[ G(s) = \frac{1}{s(s + 1)(s + 2)} \]

Designer requirement:
we want \( \zeta = 0.5 \)

This corresponds to
\( k = 1 \)
\( k_{cr} = 6 \)
\( M_G = 6 \)

System: sys
Gain: 5.97
Pole: 0.000761 + 1.41i
Damping: -0.00054
Overshoot (%): 100
Frequency (rad/sec): 1.41

Examples
3- Root locus

Roots locus exercise

\[ G(s) = \frac{1}{s(s + 2)(s^2 + 2s + 2)} \]
3- Root locus

Roots locus exercise
3- Root locus

Roots locus exercise

System: sys
Gain: 1.62
Pole: -0.373 - 0.627i
Damping: 0.511
Overshoot (%): 15.4
Frequency (rad/sec): 0.73
3- Root locus

Roots locus exercise

Control and guidance
Gain setting

Note that even though the closed-loop poles have this value of damping factor, the transitory response is not exactly sub-damped with that characteristic, because the $\zeta$ formula has been used as if it was a 2\textsuperscript{nd} order system.

However, the approximation is valid to obtain a good $\zeta$ magnitude order, the influence of poles and zeros on the response is seen in the following study.
3- Root locus

Additional pole

1. A second-order system is considered
2. A pole is added in s=-p

• system reference signal first affected by a first-order system and then by a 2\textsuperscript{nd} order one
• for a step function, signal attenuated by an exponential, which is the 2\textsuperscript{nd} order system entry

→ exit has less overshoot and it takes more time to reach its final value
3- Root locus

Additional pole

\[ f_1 = \frac{2.4}{(0.5s + 1)(s^2 + 3.2s + 5.4)} \]

\[ f_2 = \frac{2.4}{s^2 + 3.2s + 5.4} \]
3- Root locus

Additional zero

\[ f_1 = \frac{1.2(s + 2)}{s^2 + 3.2s + 5.4} \]

\[ f_2 = \frac{2.4}{s^2 + 3.2s + 5.4} \]

Zeros (negative)

- increase the initial slope,
- make the system faster so it reaches its final value earlier,
- can produce overshoot
Effect of an additional pole in the roots locus

Transfer function of a vehicle cruise-control system:

\[ G(s) = \frac{2.48}{(s + 0.06)(s + 1)(s + 3.33)} \]
A pole is added on $0$ (integrator)

$$G(s) = \frac{2.48}{s(s + 0.06)(s + 1)(s + 3.33)}$$

Roots locus moved to the right

$\rightarrow$ + unstable
3- Root locus

Effect of an additional zero in the roots locus

A zero is added in -0.12

\[ G(s) = \frac{2.48(s + 0.12)}{s(s + 0.06)(s + 1)(s + 3.33)} \]

Root locus moved to the left \( \rightarrow + \) stable, and faster
Design a controller/compensator in order to satisfy some constraints on the response of the system.

desired angle of attack $\alpha_{\text{ref}}$

speed $u_{\text{ref}}$

displacement of elevator $\delta_e$

actual angle of attack $\alpha$

speed $u$

LONITUDINAL CONTROL SYSTEM

LONITUDINAL DYNAMICS

SENSORS: INS, Anemometer

4- Controllers
4- Controllers

- **Proportional** controller: \( P: K_P \)

- **Integral** controller: \( I: \frac{K_I}{s} \)

- **Derivative** controller: \( D: K_{DS} \)

<table>
<thead>
<tr>
<th></th>
<th>( t_p )</th>
<th>( M )</th>
<th>( t_s )</th>
<th>steady-state error</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P )</td>
<td>decreases</td>
<td>increases</td>
<td>small changes</td>
<td>decreases</td>
</tr>
<tr>
<td>( I )</td>
<td>decreases</td>
<td>increases</td>
<td>increases</td>
<td>eliminates (=0)</td>
</tr>
<tr>
<td>( D )</td>
<td>small changes</td>
<td>decreases</td>
<td>decreases</td>
<td>small changes</td>
</tr>
</tbody>
</table>

- these correlations may not be exactly accurate, because \( K_P, K_I, \) and \( K_D \) are dependent of each other

- changing 1 of these variables can change the effect of the other 2
• 2 kinds of controllers improve the **transitory response**:

**Lead Compensator:**

\[ G_C(s) = \frac{s + Z_0}{s + p_0} \quad \text{with} \quad |p_0| > |Z_0| \]

adds 1 zero and 1 pole, but zero is more important: it moves the root locus to the left: improves stability (system is faster and has less overshoot)

**Proportional Derivative Compensator:**

\[ G_C(s) = K_P + K_D s \]

adds 1 zero: improves stability
• 2 kinds of controllers improve the **steady state response**:

**Lag Compensator:**

\[ G_C(s) = \frac{s + Z_0}{s + p_0} \quad \text{with} \quad |Z_0| > |p_0| \]

add 1 zero and 1 pole, but pole is more important: it moves the root locus to the right: decreases stability (system is slower and has more overshoot), but decreases the steady state error

**Proportional Integral Compensator:**

\[ G_C(s) = K_P + \frac{K_I}{s} \]
4- Controllers

- 2 kinds of controllers improve **both transitory and steady state response**:

**Lead - Lag Compensator:**

\[
G_C(s) = \frac{s + Z_0}{s + p_0} \times \frac{s + Z_1}{s + p_1}
\]

with \(|Z_0| > |p_0|\) and \(|p_1| > |Z_1|\)

**Proportional Integral Derivative (PID) Compensator:**

\[
G_C(s) = K_P + \frac{K_I}{s} + K_D s
\]
5- Frequency response

1. Fourier transforms and properties

2. Frequency response

3. Examples
1 Fourier transforms and properties

The Fourier transform of a function $x(t)$ is a function of the pulsation $\omega$:

$$F[x(t)] = X(\omega) = \int_{-\infty}^{+\infty} e^{-j\omega t} x(t) dt$$

→ It transforms a signal from the time domain to the frequency domain
1 Transforms and properties

The inverse Fourier transform recovers the original function $x(t)$:

$$x(t) = F^{-1}[X(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j\omega t} X(\omega) d\omega$$

This is true for an absolutely integrable signal:

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$$
5- Frequency response

1 Transforms and properties

Linearity:

\[ F[\alpha x(t) + \beta y(t)] = \alpha X(\omega) + \beta Y(\omega) \]

Derivation:

\[ F\left[ \frac{dx(t)}{dt} \right] = j\omega X(\omega) \]

\[ F\left[ \frac{d^n x(t)}{dt^n} \right] = (j\omega)^n X(\omega) \]
5- Frequency response

1 Transforms and properties

Additional properties

\[
F[x(at)] = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)
\]

Duality

\[
x(t) \quad \overset{F}{\rightarrow} \quad X(\omega)
\]

\[
X(t) \quad \overset{F}{\rightarrow} \quad 2\pi x(-\omega)
\]
1 Transforms and properties

**Convolution theorems**

\[
(f_1 * f_2)(t) \xrightarrow{F} F_1(\omega) \cdot F_2(\omega)
\]

\[
(f_1 \cdot f_2)(t) \xrightarrow{F} (F_1 * F_2)(\omega)
\]

where \( (f_1 * f_2)(t) = \int_{-\infty}^{+\infty} f_1(s)f_2(t-s)ds \)
Time delay

$$F[x(t \pm T)] = e^{\pm j\omega T} X(\omega)$$
5- Frequency response

1 Transforms and properties

Important pairs of transforms

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(\omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta(t-t_0)$, unit impulse</td>
<td>$e^{-j\omega t_0}$</td>
</tr>
<tr>
<td>$e^{j\omega_0 t}$</td>
<td>$2\pi \delta(\omega-\omega_0)$</td>
</tr>
<tr>
<td>$u(t)$, unit step</td>
<td>$\delta(\omega)$</td>
</tr>
<tr>
<td>$e^{-at}u(t)$</td>
<td>$\frac{1}{\omega + a}$</td>
</tr>
<tr>
<td>$\cos(2\pi \omega_0 t)$</td>
<td>$\frac{1}{2}(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$</td>
</tr>
<tr>
<td>$\sin(2\pi \omega_0 t)$</td>
<td>$\frac{1}{2j}(\delta(\omega - \omega_0) - \delta(\omega + \omega_0))$</td>
</tr>
</tbody>
</table>
In previous examples we examined the free response of an airplane with step changes in control input.

Other useful input function is the sinusoidal signal. Why?

1. Input to many physical systems takes the form or either a step change or sinusoidal signal.

2. An arbitrary function can be represented by a series of step changes or a periodic function can be decomposed by means of Fourier analysis into a series of sinusoidal waves.

→ if we know the response of a linear system to either a step or sinusoidal input then we can construct the system's response to an arbitrary input by the principle of superposition.
Example:

Examine the response of an airplane subjected to an external disturbance such as a wind gust.

Wind gust can be a sharp edged profile or a sinusoidal profile (these 2 types of gust inputs occur quite often in nature) + arbitrary gust profile can be constructed by step and sinusoidal functions.
5- Frequency response

2 Frequency response

Arbitrary wind gust profiles:
5- Frequency response

2 Frequency response

Definition of “frequency response”:

Response in steady state to a sinusoidal input

We will demonstrate that the steady state response is another sinusoidal with the same frequency
5- Frequency response

2 Frequency response

Similarity with Laplace functions with regard to the operational properties (ex: differentiation)

→ the transfer function models can be transformed from one method to the other replacing \( j\omega \) with \( s \) (or \( s \) with \( j\omega \)). (for causal signals: signals defined for positive time)
Given any system:

\[ R(s) \rightarrow G(s) \rightarrow C(s) \]

Hypothesis: **stable system**

Sinusoidal input

\[ r(t) = \sin(\omega t) \rightarrow R(s) = L[\sin(\omega t)] = \frac{\omega}{s^2 + \omega^2} \]
It can be demonstrated that the **steady state** response is:

\[ c(t) = |G(j\omega)| \sin(\omega t + \phi) \]

with \[ |G(j\omega)| = \sqrt{\text{Re}\{G(j\omega)\}^2 + \text{Im}\{G(j\omega)\}^2} \]

and \[ \phi = \text{arg}(G(j\omega)) = \arctan\left( \frac{\text{Im}\{G(j\omega)\}}{\text{Re}\{G(j\omega)\}} \right) \]
5- Frequency response

Parametric estimation

a. First-order

Frequency response: 

\[ G(j\omega) = \frac{K}{1 + \tau\omega j} = \frac{K(1 - j\omega \tau)}{1 + (\omega \tau)^2} \]

Gain: 

\[ |G(j\omega)| = \frac{K}{\sqrt{1 + (\omega \tau)^2}} \]

Delay: 

\[ \varphi = -\arctan(\tau \omega) \]
5- Frequency response

b. Second-order

\[ G(j\omega) = \frac{K\omega_n^2}{(\omega_n^2 - \omega^2) + 2\xi\omega_n j} \]

\[ |G(j\omega)| = \frac{K}{\sqrt{1 - \left(\frac{\omega}{\omega_n}\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}} \]

\[ \varphi = -\arctan \left( \frac{2\xi \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \right) \]
c. Higher-order

For a system composed by series of blocks:

\[
\begin{align*}
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qua
6- Bode diagram

1. Introduction

2. Construction rules

3. Stability
6- Bode diagrams

Goals of the Bode diagrams:

To show the frequency response characteristics in a graphical form

2 graphics for the frequency using a logarithmic scale:

• one for the **logarithm of a function magnitude** (in decibels): $|G(j\omega)|_{dB}$

• one for the **phase angle** (in degrees): $\arg\{G(j\omega)\}$

The decibel is a unit measure used to compare a certain value with a reference one. It is basically used to measure a signal power, and it is defined as:

$$|G(j\omega)|_{dB} = 10\log\left(\frac{P_{medida}}{P_{ref}}\right) = 10\log\left(\frac{|G(j\omega)|^2}{1}\right) = 20\log\{|G(j\omega)|\}$$
6- Bode diagrams

Semi-logarithmic axes: with linear scale for the magnitude or the phase, and logarithmic for the frequency.

Represents the complex transfer function adding each pole or zero effect, which compose this function (adding property of the log).
6- Bode diagrams

Gain

The gain is a factor that only modifies the magnitude and its angular value is 0°; that is, the gain value remains constant for any frequency value, because it does not depend on it.

Bode diagram for a K=50 gain
6- Bode diagrams

Integral and derivative factors

An integral factor or a pole centered in zero, has a transfer function of:

\[ G(s) = \frac{1}{s} \Rightarrow G(j\omega) = \frac{1}{j\omega} = -\frac{j}{\omega} \]

Its magnitude is, therefore:

\[ |G(j\omega)|_{dB} = 20 \log\left(\frac{1}{\omega}\right) = -20 \log(\omega) \]

For a logarithmic frequency axis: it corresponds to a straight negative line of -20 dB per decade

The phase is:

\[ \arg\{G(j\omega)\} = \arg\left\{ \frac{1}{j\omega} \right\} = \arg\left\{ -\frac{j}{\omega} \right\} = \arctan\left( \frac{-1}{0} \right) = -90^\circ \]
6- Bode diagrams

Integral and derivative factors

For a derivative factor or a zero centered in zero, the results are deduced using a similar development:

$$\left| G(j\omega) \right|_{\text{dB}} = 20 \log(\omega)$$

$$\arg\{G(j\omega)\} = 90^\circ$$
Integral and derivative factors

Bode diagrams of the derivative and the integrator
6- Bode diagrams

First-order factors: first-order pole

\[ G(j\omega) = \frac{1}{1 + j\omega\tau} \]

Its magnitude is: \[ 20 \log \left( \frac{1}{\sqrt{1 + \omega^2 \tau^2}} \right) \]

It seems more complicated, but approximations are made:

for \( \omega \ll \frac{1}{\tau} \Rightarrow \omega\tau \ll 1 \), \[ |G(j\omega)|_{\text{dB}} = -20 \log \left( \sqrt{1 + \omega^2 \tau^2} \right) \approx -20 \log(1) = 0 \]

for \( \omega \gg \frac{1}{\tau} \Rightarrow \omega\tau \gg 1 \), \[ |G(j\omega)|_{\text{dB}} = -20 \log \left( \sqrt{1 + \omega^2 \tau^2} \right) \approx -20 \log(\omega\tau) \]

- substitute the curve by its two asymptotes
- magnitude is 0 dB until it reaches the point where both asymptotes meet: \( \omega\tau = 1 \), this point is called \textit{cut frequency}
- from there: other asymptote, with a -20 dB per decade slope.
- point where approximation error is maximum corresponds to the cut frequency and the error is 3 dB.
First-order factors: first-order pole

similar for the phase, the phase real value is:

\[ \arg\{G(j\omega)\} = \arg\left\{ \frac{1}{1 + j\omega \tau} \right\} = \arg\left\{ \frac{1 - j\omega \tau}{1 + \omega^2 \tau^2} \right\} = \arctan\left\{ \frac{-\omega \tau}{1} \right\} \]

However the approximation in this case is:

\[
\begin{cases}
\text{for } \omega \ll \frac{1}{\tau} \Rightarrow \omega \tau \ll 1 & \arg\{G(j\omega)\} = 0^\circ \\
\text{for } \omega \gg \frac{1}{\tau} \Rightarrow \omega \tau \gg 1 & \arg\{G(j\omega)\} = -90^\circ 
\end{cases}
\]
6- Bode diagrams

First-order factors: first-order pole

\[ G(s) = \frac{1}{1 + s} \]

Bode diagram of a first-order system
First-order factors: first-order pole

For the study of the first-order zeros, similar development, there is only a sign change:

$$G(j\omega) = 1 + j\omega\tau$$

Magnitude:

$$20\log\left(\sqrt{1 + \omega^2\tau^2}\right) = 10\log\left(1 + \omega^2\tau^2\right)$$

Angular contribution:

$$\arg\{G(j\omega)\} = \arg\{1 + j\omega\tau\} = \arctan\left\{\frac{\omega\tau}{1}\right\}$$


6- Bode diagrams

First-order factors: first-order pole

Example:

\[ G(s) = \frac{10}{2s + 1} \]
6- Bode diagrams

Second-order factors

\[
G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \Rightarrow G(j\omega) = \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n \omega + \omega_n^2} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j \left(2\zeta \frac{\omega}{\omega_n}\right)}
\]

The magnitude is:

\[
|G(j\omega)| = 20 \log \sqrt{\frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}} = -20 \log \sqrt{1 - \left(\frac{\omega}{\omega_n}\right)^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}
\]
Second-order factors

\[ |G(j\omega)| = -20\log\left(1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2\xi\frac{\omega}{\omega_n}\right)^2 \]

it can be approximated this way:

for \( \frac{\omega}{\omega_n} \ll 1 \), \( |G(j\omega)| \approx -20\log(1) = 0 \text{ dB} \)

for \( \frac{\omega}{\omega_n} \gg 1 \), \( |G(j\omega)| \approx -20\log\left(\frac{\omega}{\omega_n}\right)^2 \)

For low frequencies: straight line at 0dB

For high frequencies: straight line with a –40 dB per decade slope.

Both asymptotes cross on \( \omega = \omega_n \).

However, in the second-order poles a resonance effect can appear.

In the frequency domain the resonance is shown as a peak close to the cut frequency; the resonance peak value is conditioned to the \( \zeta \) value.
6- Bode diagrams

Second-order factors

\[ 2\zeta \left( \frac{\omega}{\omega_n} \right) \]

For the phase: \( \tan \Phi = -\frac{2\zeta \left( \frac{\omega}{\omega_n} \right)}{1 - \left( \frac{\omega}{\omega_n} \right)} \)

For \( \frac{\omega}{\omega_n} << 1 \), generally \( \frac{\omega}{\omega_n} < 0.2 \), \( \Phi \approx -\arctan(0) = 0\degree \)

For \( \frac{\omega}{\omega_n} >> 1 \), generally \( \frac{\omega}{\omega_n} > 5 \), \( \Phi \approx -\arctan(0^-) = -180\degree \)

For \( \frac{\omega}{\omega_n} = 1 \), \( \Phi = -\arctan(\infty) = -90\degree \)

The phase graphic form depends also on \( \zeta \)
6- Bode diagrams

Second-order factors

Bode diagram of a second-order pole for different $\zeta$ values
6- Bode diagrams

Second-order factors

Example:

\[ G(s) = \frac{2.5}{s^2 + 3s + 25} \]
6- Bode diagrams

Stability condition

Notation: \( G = 0 \) for \( \omega = \omega_{G0} \) and \( \Phi = -180^\circ \) for \( \omega = \omega_{G-180^\circ} \)

The Bode diagram in open loop is studied

Stability condition:

If for \( \omega = \omega_{G0} \), \( \Phi > -180^\circ \)  
And for \( \omega = \omega_{-180^\circ} \), \( G < 0 \)

THEN the system is STABLE in closed loop

If for \( \omega = \omega_{G0} \), \( \Phi < -180^\circ \)  
Or for \( \omega = \omega_{-180^\circ} \), \( G > 0 \)

THEN the system is UNSTABLE in closed loop
6- Bode diagrams

Stability condition

Bode Diagram

Magnitude (dB)

Phase (deg)

Frequency (rad/sec)

STABLE
6- Bode diagrams

Stability condition

Bode Diagram

Magnitude (dB)

Phase (deg)

Frequency (rad/sec)

UNSTABLE
6- Bode diagrams

Stability margins

Common values:
- Minimum gain margin: 10 to 12 dB,
- Minimum phase margin: 45 to 50°
6- Bode diagrams

Stability margins

\[ G(s) = \frac{1}{s(s + 1)(s + 2)} \]
6- Bode diagrams

Stability margins

\[ G(s) = \frac{1}{s(s+1)(s+2)} \]

- \( M_G = 14.5 \text{dB} \)
- with roots locus:
  - \( M_G = 6 \),
  - and note that:
  - \( 20 \log(6) = 15.5 \text{dB} \)
6- Bode diagrams

Stability margins

\[
G(s) = \frac{1}{s(s+1)(s+2)}
\]

\[M_\phi = 51^\circ\]
6- Bode diagrams

Stability margins

Transfer function of a vehicle cruise-control system

\[ G(s) = \frac{2.48}{(s + 0.06)(s + 1)(s + 3.33)} \]
A pole is added on 0 (integrator): Bode diagram shifted downward: + unstable
Stability margins

A zero is added in -0.12: Bode diagram shifted upward: + stable
REFERENCES


