SYNCHRONIZATION TECHNIQUES FOR DIGITAL DEMODULATION

TRANSMISSION METHODS

- ***COHERENT MODULATION*** requires knowledge of:
    - Carrier Phase and Frequency
    - Timing

- PSK, APK, QAM
- CPM

- ***NON-COHERENT MODULATION*** requires knowledge of:
    - Timing

- Power penalty and Spectral inefficiency

DIGITAL SYNCHRONIZATION CONFIGURATIONS

- Two digital synchronization schemes:
  - **CLOSED LOOP** or FEEDBACK
    - Used in CONTINUOUS mode (SCPC)
    - An error signal is used to drive the previous stage.
  - **OPEN-LOOP** or FEEDFORWARD
    - Used in BURST mode (TDMA)
    - Very few symbols to synchronize

- Estimation symbols dependence:
  - **DA Estimation** (Data Aided): use a known symbol sequence
  - **DD Estimation** (Decision Directed): use symbols decided by the demodulator
  - **NDA Estimation** (Non-DA): eliminate the symbol dependence

Impact of Phase Error Bias and Variance (BPSK)

- Lets consider a residual carrier phase error:

$$I' + jQ' = A(I + jQ)e^{-j\theta_\varepsilon} + \left(n_i + jn_q\right)$$

- If the error is small, we can consider that:

$$e^{-j\theta_\varepsilon} \approx 1 - j\theta_\varepsilon - \frac{\theta_\varepsilon^2}{2!} + j\frac{\theta_\varepsilon^3}{3!} + ...$$

- For a BPSK signal (Q=0), decision are done over the In-Phase component:

$$I' \approx AI\left(1 - \frac{\theta_\varepsilon^2}{2}\right) + n_i \Rightarrow \hat{I} = sign[I'] = sign\left[AI\left(1 - \frac{\theta_\varepsilon^2}{2}\right) + n_i\right]$$

$$Q' \approx -AI\theta_\varepsilon + n_q \quad (irrelevant)$$

- Thus, the signal and the noise terms in the decision are:

$$\boxed{\begin{aligned} &\text{signal term} = AI \\ &\text{noise term} = -AI\frac{\theta_\varepsilon^2}{2} + n_i \end{aligned}}$$

Phase Error Variance-BPSK

- At this step we consider that there is no bias in the carrier error and it is modeled as a random AWGN variable:

$$\theta_\varepsilon \approx N(0;\sigma_\theta^2) = \begin{cases} E[\theta_\varepsilon] = 0 \\ E[\theta_\varepsilon^2] = \sigma_\theta^2 \Rightarrow E[\theta_\varepsilon^4] = 3(\sigma_\theta^2)^2 \end{cases}$$

- The energy per bit (E_b) at the matched filter output will be:

$$E_b = A^2 E[I^2] T_b = A^2 T_b$$

- The noise power will be given by:

$$\sigma_{n+\theta}^2 = A^2 E[I^2] \frac{E[\theta_\varepsilon^4]}{4} + E[n_i^2] = A^2 \frac{E[\theta_\varepsilon^4]}{4} + \frac{N_0}{T_b}$$

and the carrier variance increments the global detection noise.

- The equivalent noise density becomes:

$$N_0^{n+\theta} = \frac{\sigma_{n+\theta}^2}{r_b} = A^2 \frac{E[\theta_\varepsilon^4]}{4} T_b + N_0$$

Phase Error Variance-BPSK (II)

- Thus, the effective Eb/No becomes:

$$\frac{E_b}{N_0^{n+\theta}} = \frac{E_b}{A^2 \frac{E\left[\theta_\varepsilon^4\right]}{4} T_b + N_0} = \frac{\left(E_b / N_0\right)}{\frac{E\left[\theta_\varepsilon^4\right]}{4}\left(E_b / N_0\right) + 1}$$

- And finally:

$$\boxed{\frac{E_b}{N_0^{n+\theta}} = \frac{\left(E_b / N_0\right)_{nominal}}{\frac{3}{4}\left(\sigma_\varepsilon^2\right)^2 \left(E_b / N_0\right)_{nominal} + 1}}$$

- It is easy to see that the last equation is also useful when there is an error BIAS or a carrier synchronization tracking error:

$$\frac{E_b}{N_0^{n+\theta}} = \frac{\left(E_b / N_0\right)_{nominal} (1 - bias_\theta^2)^2}{\frac{3}{4}\left(\sigma_\varepsilon^2\right)^2 \left(E_b / N_0\right)_{nominal} + 1}$$

INTRODUCTION TO ESTIMATION THEORY

Obtain the parameters θ which belong to an infinite alphabet (otherwise we deal with a detection problem)

$$\hat{\theta} = g\left(x[0], x[1], \ldots, x[N-1]\right) = g(\mathbf{x})$$

E.g.:
* Synchronization in coherent communication receivers,
* Position estimation in radar and sonar
* Speech recognition

...

* Data has to be modelled mathematically (pdf, membership functions,...)

* θ: deterministic parameter but unknown — **Classic estimation theory**

or

random parameter — **Bayesian estimation theory**

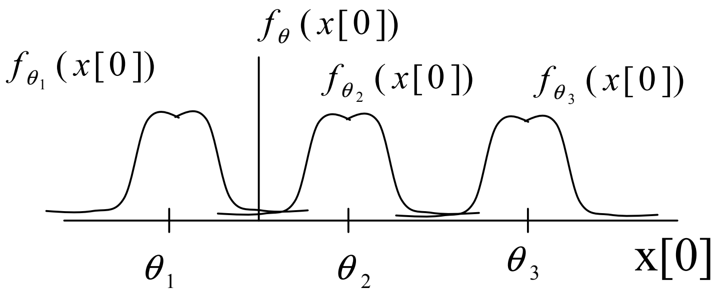
θ : deterministic (Classic theory)

E.g.: $N=1$ $$f_\theta(x[0]) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2\sigma^2}(x[0]-\theta)^2\right)$$

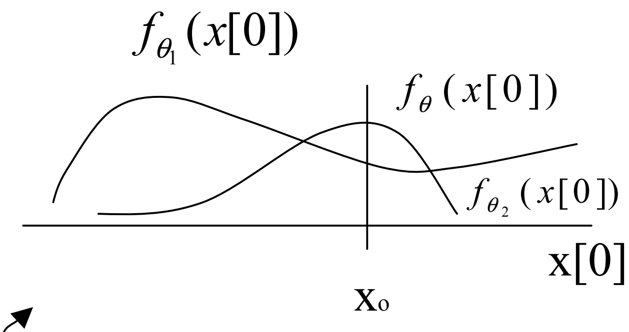
R.V. Unknown parameter

$f_\theta(x[0])$: family of pdf's. parameterized pdf

Because the value of θ affects the probability of x[0], we should be able to infer the value of θ from the observed value of x[0]. In other words, the measurement x[0] brings information about θ through the probability law



in general



x_o is more probable to be observed if $\theta=\theta_2$ than if $\theta=\theta_1$

For the general case of having a collection of N real data

$$\mathbf{x} = \left[x[0]\; x[1] \,\Lambda\;\; x[N-1] \right]^T$$

and a vector of p parameters $\boldsymbol{\theta}$

$$f_\theta(\mathbf{x}) = \frac{1}{\left(2\pi\sigma^2\right)^{N/2}} \exp\left(-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} \left(x[n] - f(\boldsymbol{\theta})\right)^2 \right)$$

θ : random variable (Bayesian Philosophy)

A priori knowledge of θ can be introduced by considering θ as a r.v. whose pdf is known.

The use of prior knowledge will lead to a more accurate estimation.

The parameter we are attempting to estimate is then viewed as a realization of the random variable θ. As such, the data are describe by the joint pdf.

$$f(x[0],\theta) = f(\mathbf{x}/\theta)\ f(\theta)$$

Marginal or a priori pdf

Known pdf.
It is not a family of pdf's as $f_\theta(x[0])$ was

An estimator may be thought of as a rule that assigns a value to θ for each realization of **x**

The resulting estimator is "optimum" in average or referring the a priori pdf that has been considered.

Wiener and Kalman filtering follow the Bayesian philosophy.

Assessing Estimator Performance

Because an estimator is a function of the data, which is a r.v., it too is a r.v. Therefore to assess performance we must do so statistically.

Mean Square Error (MSE): $MSE = E\left\{\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right)^2\right\} = \left(E\left\{\hat{\boldsymbol{\theta}}\right\} - \boldsymbol{\theta}\right)^2 + E\left\{\left(\hat{\boldsymbol{\theta}} - E\left\{\hat{\boldsymbol{\theta}}\right\}\right)^2\right\}$

where $E\{\hat{\boldsymbol{\theta}}\} = \left[E\{\hat{\theta}_1\} \;\; E\{\hat{\theta}_2\} \;\; \Lambda \;\; E\{\hat{\theta}_P\}\right]^T$

Bias: $b(\hat{\boldsymbol{\theta}}) = E\{\hat{\boldsymbol{\theta}}\} - \boldsymbol{\theta}$ Deviation from the correct value

Variance: $\mathrm{var}(\hat{\boldsymbol{\theta}}) = E\left\{(\hat{\boldsymbol{\theta}} - E\{\hat{\boldsymbol{\theta}}\})^2\right\}$ It is the MSE when $b(\hat{\boldsymbol{\theta}}) = 0$

- In the classical or deterministic approach:

$$MSE = E\left\{\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right)^2\right\} = \int \left(\hat{\boldsymbol{\theta}}(\mathbf{x}) - \boldsymbol{\theta}\right)^2 f_\theta(\mathbf{x})\, d\mathbf{x}$$

- In the Bayesian approach:

$$MSE = E\left\{\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right)^2\right\} = \int\int \left(\hat{\boldsymbol{\theta}}(\mathbf{x}) - \boldsymbol{\theta}\right)^2 f(\mathbf{x};\theta)\, d\mathbf{x}\, d\theta$$

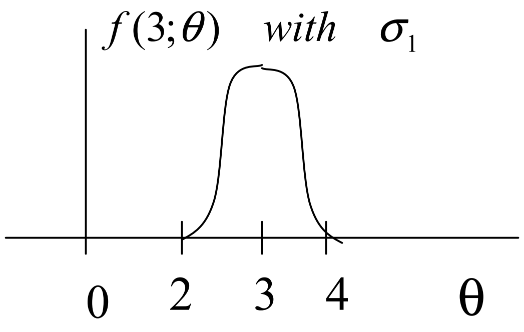
Cramer Rao Lower Bound

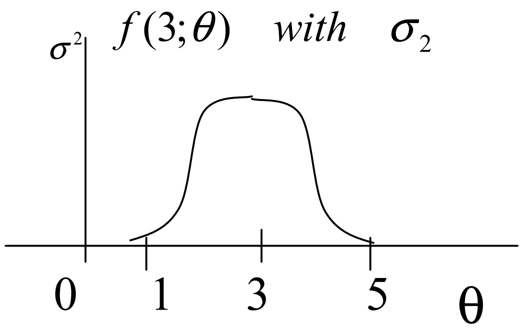
Estimator accuracy considerations

Since all our information is embodied in the observed data and the underlying pdf. for that data, it is not surprising that the estimation accuracy depends directly on the pdf. In general, the more the pdf is influenced by the unknown parameter, the better we should be able to estimate it.

When the pdf is viewed as a function of the unknown parameter (with **x** fixed) it is termed the *likelihood function*.

Example: $$f(3;\theta)=\frac{1}{\sqrt{2\pi}\sigma_i}\exp\left(-\frac{1}{2\sigma_i^2}(3-\theta)^2\right)$$





Intuitively, the "sharpness" of the likelihood function determines how accurately we can estimate the unknown parameter. To quantify this notion observe that the sharpness is effectively measured by the negative of the second derivative of the logarithm of the likelihood function at its peak. This is the curvature of the log-likelihood function.

$$\boxed{\textbf{"sharpness":} - E\left\{\frac{\partial^2 \ln f_\theta(\mathbf{x})}{\partial \theta^2}\right\}}$$

Evaluated in the actual value of θ

Regularity condition $E\left[\frac{\partial \ln\ f_\theta(\mathbf{x})}{\partial\theta}\right]=0$

Score function

$$\int \frac{\partial \ln\ f_\theta(\mathbf{x})}{\partial\theta} f_\theta(\mathbf{x})d\mathbf{x} = \int \frac{1}{f_\theta(\mathbf{x})}\frac{\partial f_\theta(\mathbf{x})}{\partial\theta} f_\theta(\mathbf{x})d\mathbf{x} = \frac{\partial}{\partial\theta}\int f_\theta(\mathbf{x})d\mathbf{x} = 0$$

Considering an unbiased estimator

$$E(\hat{\theta})=\theta \rightarrow E(\hat{\theta}-\theta)=0 \rightarrow \int(\hat{\theta}-\theta)\ \ f_\theta(\mathbf{x})\ \ d\mathbf{x}=0$$

Differentiating with respect to θ and interchanging the partial differentiation and integration

$$\int(\hat{\theta}-\theta)\frac{\partial \ln f_\theta(\mathbf{x})}{\partial\theta}\quad f_\theta(\mathbf{x})\quad d\mathbf{x}=1$$

Applying the Cauchy-Schwarz inequality

$$\left[\int w(\mathbf{x})g(\mathbf{x})h(\mathbf{x})\quad d\mathbf{x}\right]^2 \le \int w(\mathbf{x})g^2(\mathbf{x})d\mathbf{x}\int w(\mathbf{x})h^2(\mathbf{x})d\mathbf{x}$$

$$E\left\{\frac{\partial \ln f_\theta(\mathbf{x})}{\partial\theta}(\hat{\theta}-\theta)\right\}^2 \leq E\left\{(\hat{\theta}-\theta)^2\right\}E\left\{\left(\frac{\partial \ln f_\theta(\mathbf{x})}{\partial\theta}\right)^2\right\}$$

It is obtained that the variance of any unbiased estimator $\hat{\theta}$ must satisfy

$$\boxed{\operatorname{var}(\hat{\theta}) \geq \frac{1}{-E\left\{\frac{\partial^2 \ln f_\theta(\mathbf{x})}{\partial\theta^2}\right\}} = \frac{1}{E\left\{\left(\frac{\partial \ln f_\theta(\mathbf{x})}{\partial\theta}\right)^2\right\}} =}$$

CRLB

$$\frac{\partial}{\partial\theta}\left(\frac{\partial \ln f_\theta(\mathbf{x})}{\partial\theta}\right) = \frac{f_\theta(\mathbf{x})\frac{\partial}{\partial\theta}\left(\frac{\partial f_\theta(\mathbf{x})}{\partial\theta}\right) - \frac{\partial}{\partial\theta}f_\theta(\mathbf{x})\frac{\partial f_\theta(\mathbf{x})}{\partial\theta}}{f_\theta^2(\mathbf{x})} = \frac{\frac{\partial}{\partial\theta}\left(\frac{\partial f_\theta(\mathbf{x})}{\partial\theta}\right)}{f_\theta(\mathbf{x})} - \left(\frac{\partial \ln f_\theta(\mathbf{x})}{\partial\theta}\right)^2$$

$$E\left[\frac{\frac{\partial}{\partial\theta}\left(\frac{\partial f_\theta(\mathbf{x})}{\partial\theta}\right)}{f_\theta(\mathbf{x})}\right] = \int \frac{\frac{\partial}{\partial\theta}\left(\frac{\partial f_\theta(\mathbf{x})}{\partial\theta}\right)}{f_\theta(\mathbf{x})} f_\theta(\mathbf{x})d\mathbf{x} = \frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\theta}\int f_\theta(\mathbf{x})d\mathbf{x}\right) = 0$$

Extension to a vector parameter case $\boldsymbol{\theta}$

$$-E\left\{\frac{\partial^2 \ln f_\theta(\mathbf{x})}{\partial \boldsymbol{\theta}^2}\right\} = E\left\{\left(\frac{\partial \ln f_\theta(\mathbf{x})}{\partial \boldsymbol{\theta}}\right)^2\right\} = \mathbf{J}$$ Fisher information matrix

$$\frac{\partial^2 \ln f_\theta(\mathbf{x})}{\partial \boldsymbol{\theta}^2}$$ Stochastic Fisher information matrix

$$\boxed{\operatorname{var}(\hat{\theta}_i) \geq [\mathbf{J}^{-1}]_{ii}}$$

The more information is available less is the variance

Modified Cramer Rao Lower Bound

Estimation of a single element of θ generically denoted by λ. The other parameters are collected in a random vector **u** having a known probability density function.

To compute the CRB we need: $f(\mathbf{x}\,|\,\lambda)=\int f(\mathbf{x}\,|\,\mathbf{u},\lambda)f(\mathbf{u})d\mathbf{u}$

$$CRB(\lambda)=\frac{1}{E\left\{\left(\dfrac{\partial \ln f(\mathbf{x}\,|\,\lambda)}{\partial\lambda}\right)^2\right\}}$$

$$\boxed{MCRB(\lambda)=\frac{1}{E_{\mathbf{x},\mathbf{u}}\left\{\left(\dfrac{\partial \ln f(\mathbf{x}\,|\,\mathbf{u},\lambda)}{\partial\lambda}\right)^2\right\}}}$$

$$MCRB(\lambda)\geq CRB(\lambda)$$