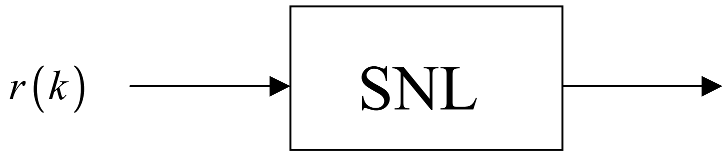
OPEN LOOP PHASE SYNCHONIZERS

Non Data Aided

M-PSK $\quad r(k) = A_0 e^{j\left(w_0 k + \theta_0 + \frac{2\pi}{M} m\right)} \quad m = 0, 1, \ldots M-1$



$$A_0^M e^{j\left(M w_0 k + M\theta_0 + \frac{2\pi}{M} Mm\right)} = A_0' e^{j\left(w_0' k + \theta_0'\right)}$$

$$w_0' = M w_0$$

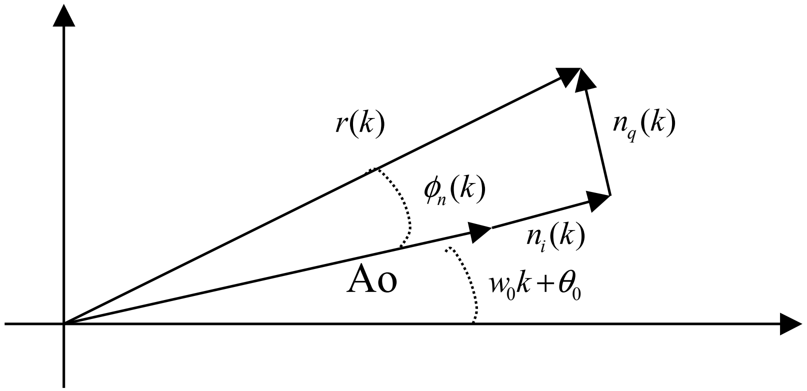
$$\theta_0' = M\theta_0 \quad \text{mod} \quad 2\pi$$

Viterbi & Viterbi

$$r(k) = a(k) + jb(k) = \rho(k) e^{j\phi(k)}$$

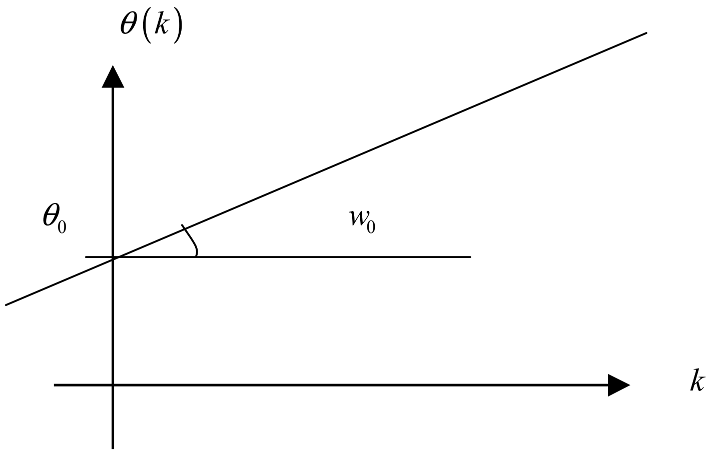
$$\rho(k) = \sqrt{a(k)^2 + b(k)^2} \quad \phi(k) = \tan^{-1}\left(\frac{b(k)}{a(k)}\right)$$

$$r'(k) = a'(k) + jb'(k) = F\left(\rho(k)\right) e^{jm\phi(k)}$$



$$r(k) = A_0 e^{j(w_0 k + \theta_0)} + w(k) \cong A_0' e^{j\left(w_0 k + \theta_0 + \frac{n_q}{A_0}\right)}$$

$$\phi_n(k) = \arctan\left[\frac{n_q(k)}{A_0 + n_i(k)}\right] \cong \frac{n_q(k)}{A_0}$$



$$\hat{\theta}(k) = \arg r(k) = w_0 k + \theta_0 + \frac{n_q}{A_0}$$

$$\theta(k) = w_0 k + \theta_0$$

2N+1 measurements:

$$\hat{\boldsymbol{\theta}}(k) = \begin{bmatrix} \hat{\theta}(-N) \\ \hat{\theta}(-N+1) \\ \vdots \\ \hat{\theta}(N) \end{bmatrix} \qquad \mathbf{N}(k) = \begin{bmatrix} -N & 1 \\ -N+1 & 1 \\ \vdots & \vdots \\ N & 1 \end{bmatrix} \qquad \mathbf{a} = \begin{bmatrix} w_0 \\ \theta_0 \end{bmatrix}$$

Estimation Error: $\mathbf{e} = \hat{\boldsymbol{\theta}} - \mathbf{N}\mathbf{a}$

$$\min_{\mathbf{a}} \left\| \mathbf{e} \right\|^2 = \min_{\mathbf{a}} \left\| \hat{\boldsymbol{\theta}} - \mathbf{N}\mathbf{a} \right\|^2 \Rightarrow \mathbf{a} = \left(\mathbf{N}^T \mathbf{N} \right)^{-1} \mathbf{N}^T \hat{\boldsymbol{\theta}}$$

$$\boxed{\hat{\theta}_o = \frac{1}{2N+1} \sum_{k=-N}^{N} \hat{\theta}(k)}$$

Uniform averager of unwrapped phases

$$\boxed{\hat{w}_o = \frac{3}{N(N+1)(2N+1)} \sum_{k=-N}^{N} k\hat{\theta}(k)}$$

FREQUENCY ESTIMATORS

1. The offset is much smaller than 1/T

    The receiver is operating in steady-state conditions.

    Timing is recovered first (even with frequency errors as large as 20% of the symbol rate).

    DA, DD, NDA

2. The offset is on the order of the symbol rate 1/T

    Initial frequency acquisitions in satellite communication systems.

    Data symbols, carrier phase and timing are unknown.

    NDA

1. The offset is much smaller than 1/T

- DA Frequency Estimation

  Data symbols are known.

  Timing is ideal.

  The frequency offset is less than 10% of the symbol rate.

  Kay, Fitz and Luise-Reggiannini methods.

- DD Frequency Estimation

  DPSK

- NDA Frequency Estimation

  Data symbols are unknown.

  Open-loop or closed-loop methods

KAY METHOD

$$y(k) = c_k e^{j[2\pi\upsilon(kT+\tau)+\theta]} + n(k) \Longrightarrow z(k) = y(k)c_k^*$$

$$z(k) = e^{j[2\pi\upsilon(kT+\tau)+\theta]} + n(k) = \rho(k)e^{j[2\pi\upsilon(kT+\tau)+\theta+\phi(k)]}$$

$$\arg\{z(k)z^*(k-1)\} = 2\pi\upsilon T + \phi(k) - \phi(k-1)$$

Observation sequence: $\alpha(k) = \arg\{z(k)z^*(k-1)\}$

ML estimator:

$$\hat{\upsilon} = \frac{1}{2\pi T}\sum_{k=1}^{L_0-1}\gamma(k)\arg\{z(k)z^*(k-1)\}$$

$$\gamma(k) = \frac{3}{2}\frac{L_0}{L_0^2-1}\left[1-\left(\frac{2k-L_0}{L_0}\right)^2\right] \quad k = 1,2,\ldots L_0-1$$

FITZ METHOD

$$R(m) = \frac{1}{L_0 - m} \sum_{k=m}^{L_0 - 1} z(k) z^*(k-m) \quad 1 \le m \le L_0 - 1$$

$$R(m) = e^{j2\pi m\upsilon T} + n'(k) \quad 1 \le m \le L_0 - 1$$

$$e(m) \text{ @} \arg\left\{R(m)\right\} - 2\pi m \upsilon T \quad \bmod 2\pi$$

$$\frac{1}{N}\sum_{m=1}^{N} e(m) = \frac{1}{N}\sum_{m=1}^{N} \arg\left\{R(m)\right\} - \pi(N+1)\upsilon T \quad N < \frac{1}{2\left|\upsilon_{\max}\right| T}$$

$$\boxed{\hat{\upsilon} = \frac{1}{\pi N (N+1) T} \sum_{m=1}^{N} \arg\left\{R(m)\right\}}$$

LUISE AND REGGIANNINI METHOD

$$R(m) = e^{j2\pi m\upsilon T} + n'(k) \quad 1 \le m \le N$$

$$\frac{1}{N}\sum_{m=1}^{N} R(m) = \frac{1}{N}\sum_{m=1}^{N} e^{j2\pi m\upsilon T} + \frac{1}{N}\sum_{m=1}^{N} n'(k)$$

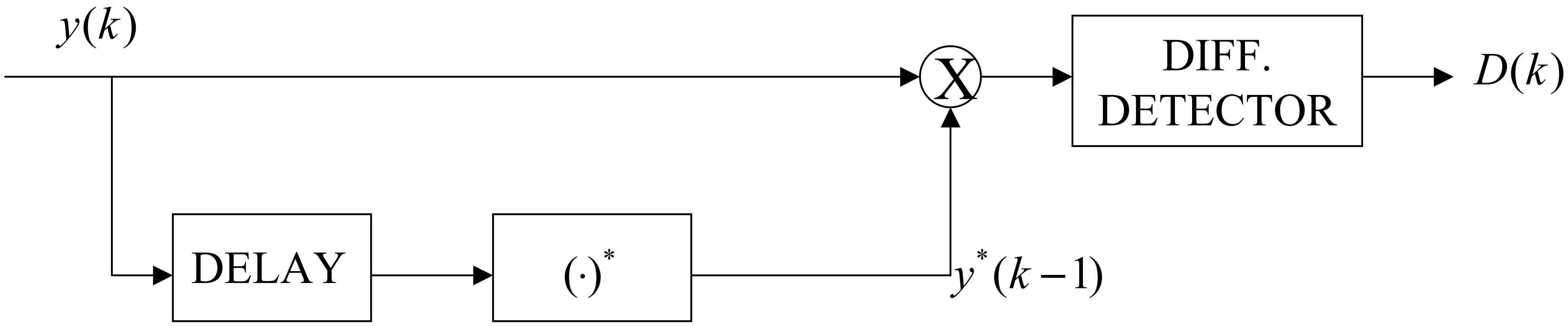
$$\sum_{m=1}^{N} e^{j2\pi m\upsilon T} \approx \sum_{m=1}^{N} R(m)$$

$$\sum_{m=1}^{N} e^{j2\pi m\upsilon T} = \frac{\sin \pi N\upsilon T}{\sin \pi\upsilon T} e^{j\pi(N+1)\upsilon T} \quad \Rightarrow \quad \upsilon = \frac{1}{\pi(N+1)T} arg\left\{\sum_{m=1}^{N} e^{j2\pi m\upsilon T}\right\}$$

$$|\upsilon| \le \frac{1}{NT}$$

$$\boxed{\hat{\upsilon} = \frac{1}{\pi(N+1)T}\arg\left\{\sum_{m=1}^{N} R(m)\right\}}$$

DD FREQUENCY RECOVERY WITH DPSK



Assuming reliable decisions: $D(k) \approx c_k c_{k-1}^*$ $\qquad z(k) = y(k) c_k^*$

$$z(k) z^*(k-1) = y(k) y^*(k-1) \left(c_k c_{k-1}^*\right)^* = y(k) y^*(k-1) D^*(k)$$

$$R(1) = \frac{1}{L_0 - 1} \sum_{k=1}^{L_0 - 1} z(k) z^*(k-1) = \frac{1}{L_0 - 1} \sum_{k=1}^{L_0 - 1} y(k) y^*(k-1) D^*(k)$$

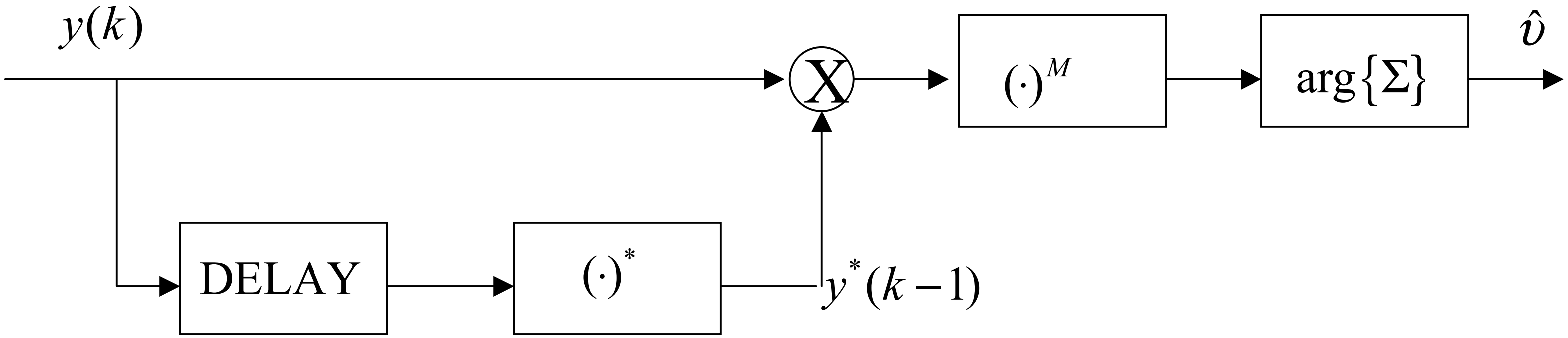
$$\boxed{\hat{\upsilon} = \frac{1}{2\pi T} arg \left\{ \sum_{k=1}^{L_0 - 1} y(k) y^*(k-1) D^*(k) \right\}}$$

NDA OPEN LOOP ALGORITHM

$$y(k) = c_k e^{j[2\pi\upsilon(kT+\tau)+\theta]} + n(k)$$

M-PSK: $c_k^M = 1 \longrightarrow y^M(k) = e^{j[2M\pi\upsilon(kT+\tau)+M\theta]} + n'(k)$

$$\left[y(k)\,y^*(k-1)\right]^M = e^{j2M\pi\upsilon T} + n''(k)$$



$$\frac{1}{L_0-1}\sum_{k=1}^{L_0-1}\left[y(k)\,y^*(k-1)\right]^M = e^{j2M\pi\upsilon T} + \frac{1}{L_0-1}\sum_{k=1}^{L_0-1} n''(k)$$

$$\hat{\upsilon} = \frac{1}{2M\pi T}\,arg\left\{\sum_{k=1}^{L_0-1}\left[y(k)\,y^*(k-1)\right]^M\right\}$$

CLOSE-LOOP WITH NO TIMING INFORMATION

$$s(t) = e^{j(2\pi \upsilon t + \theta)} \sum_i c_i g(t - iT - \tau)$$

υ is an unknown constant, in the range $\pm 1/T$

θ is a random variable uniformly distributed over $[0,2\pi)$

τ is a random variable uniformly distributed over $[0,T)$

$\{c_i\}$ are zero-mean independent variables with the following second-order moments:

$$E\{c_i c_k^*\} = \begin{cases} C & i = k \\ 0 & elsewhere \end{cases}$$

θ, τ, and $\{c_i\}$ are independent of each other

Likelihood function:

$$\Lambda\left(\mathbf{r} \mid \tilde{\nu}, \tilde{\theta}, \tilde{\tau}, \tilde{\alpha}\right) = \exp\left\{ \frac{1}{N_0} \int_0^{T_0} \mathrm{Re}\left[r(t)\, \tilde{s}^*(t) \right] dt - \frac{1}{2N_0} \int_0^{T_0} \left| \tilde{s}(t) \right|^2 dt \right\}$$

$$\tilde{s}(t) = e^{j\left(2\pi \tilde{\nu} t + \tilde{\theta}\right)} \sum_i \tilde{\alpha}_i\, g\left(t - iT - \tilde{\tau}\right)$$

Assuming low SNR, such that the expansion of the exponential into a power series can be truncated to the quadratic term:

$$X_{rs} \triangleq \int_0^{T_0} \mathrm{Re}\left[r(t)\, \tilde{s}^*(t) \right] dt \qquad X_{ss} \triangleq \int_0^{T_0} \left| \tilde{s}(t) \right|^2 dt$$

$$\Lambda\left(\mathbf{r} \mid \tilde{\nu}, \tilde{\theta}, \tilde{\tau}, \tilde{\alpha}\right) \approx 1 + \frac{1}{2N_0}\left(2X_{rs} - X_{ss}\right) + \frac{1}{8N_0^2}\left(2X_{rs} - X_{ss}\right)^2$$

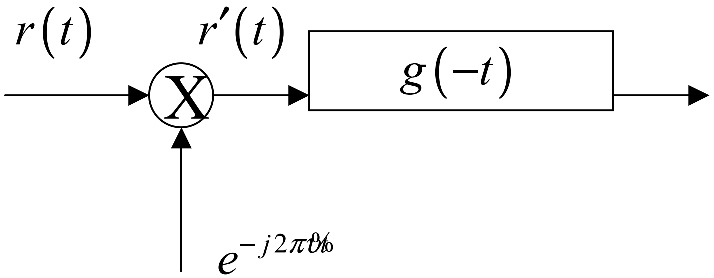
$\Lambda(\mathbf{r} \mid \tilde{\nu})$ is the expectation of $\Lambda\left(\mathbf{r} \mid \tilde{\nu}, \tilde{\theta}, \tilde{\tau}, \tilde{\alpha}\right)$ with respect to

$\tilde{\mathbf{u}} \triangleq \{\tilde{\theta}, \tilde{\tau}, \tilde{\alpha}\}$ vector of unwanted parameters

$$\Lambda(\mathbf{r} \mid \tilde{\nu}) = A_1 E_{\tilde{\mathbf{u}}}\left\{X_{rs}^2\right\} + A_2$$

Constants independent of $\tilde{\nu}$ $\Longrightarrow$ $\Lambda'(\mathbf{r} \mid \tilde{\nu}) = E_{\tilde{\mathbf{u}}}\left\{X_{rs}^2\right\}$

$$\int_0^{T_0} r(t)\,\tilde{s}^*(t)\,dt \approx e^{-j\tilde{\theta}} \sum_{i=0}^{L_0-1} \tilde{c}_i^*\, x(iT + \tilde{\tau}) \qquad L_0 \triangleq T_0 / T$$



$$x(t) \triangleq \int_{-\infty}^{\infty} r(\xi)\, e^{-j2\pi\tilde{\nu}\xi}\, g(\xi - t)\, d\xi$$

$$X_{rs} \triangleq \int_0^{T_0} \mathrm{Re}\left[r(t)\tilde{s}^*(t) \right] dt = \frac{1}{2} e^{-j\tilde{\theta}} \sum_{i=0}^{L_0-1} \tilde{a}_i^* x\left(iT+\tilde{\tau}\right) + \frac{1}{2} e^{j\tilde{\theta}} \sum_{i=0}^{L_0-1} \tilde{a}_i x^*\left(iT+\tilde{\tau}\right)$$

$$\begin{aligned} X_{rs}^2 = & \frac{1}{2} \sum_{i=0}^{L_0-1} \sum_{k=0}^{L_0-1} \tilde{a}_i^* \tilde{a}_k x\left(iT+\tilde{\tau}\right) x^*\left(kT+\tilde{\tau}\right) + \\ & \frac{1}{4} e^{-j2\tilde{\theta}} \sum_{i=0}^{L_0-1} \sum_{k=0}^{L_0-1} \tilde{a}_i^* \tilde{a}_k^* x\left(iT+\tilde{\tau}\right) x\left(kT+\tilde{\tau}\right) + \\ & \frac{1}{4} e^{j2\tilde{\theta}} \sum_{i=0}^{L_0-1} \sum_{k=0}^{L_0-1} \tilde{a}_i \tilde{a}_k x^*\left(iT+\tilde{\tau}\right) x^*\left(kT+\tilde{\tau}\right) \end{aligned}$$

$$E_{\tilde{\theta},\tilde{a}}\left\{X_{rs}^2\right\} = \frac{C}{2} \sum_{i=0}^{L_0-1} \left| x\left(iT+\tilde{\tau}\right) \right|^2 \qquad \Lambda'\left(\mathbf{r} \mid \tilde{\tau}\right) = \frac{C}{2T} \sum_{i=0}^{L_0-1} \int_0^T \left| x\left(iT+\tilde{\tau}\right) \right|^2 d\tilde{\tau}$$

$$\boxed{\Lambda''\left(\mathbf{r} \mid \tilde{\tau}\right) = \int_0^{T_0} \left| x(t) \right|^2 dt}$$

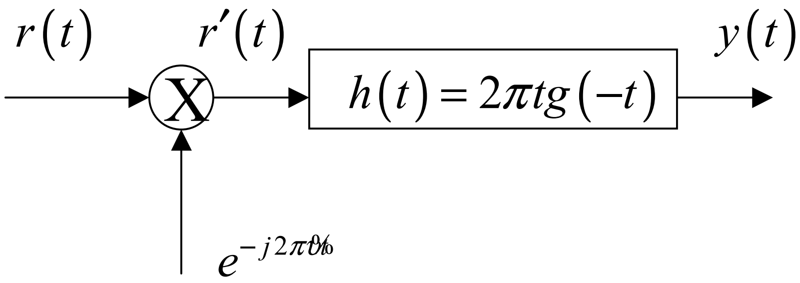
Energy of the matched filter output

$$\frac{d\Lambda''(\mathbf{r}\mid\tilde{\vartheta})}{d\tilde{\vartheta}} = 2\int_0^{T_0} \operatorname{Re}\left\{ x(t)\frac{\partial x^*(t)}{\partial\tilde{\vartheta}} \right\} dt$$

$$\frac{\partial x(t)}{\partial\tilde{\vartheta}} = jy(t) - j2\pi t x(t)$$

$$y(t) \triangleq \int_{-\infty}^{\infty} r(\xi) e^{-j2\pi\tilde{\vartheta}\xi}\, 2\pi(t-\xi)\, g(\xi - t)\, d\xi$$

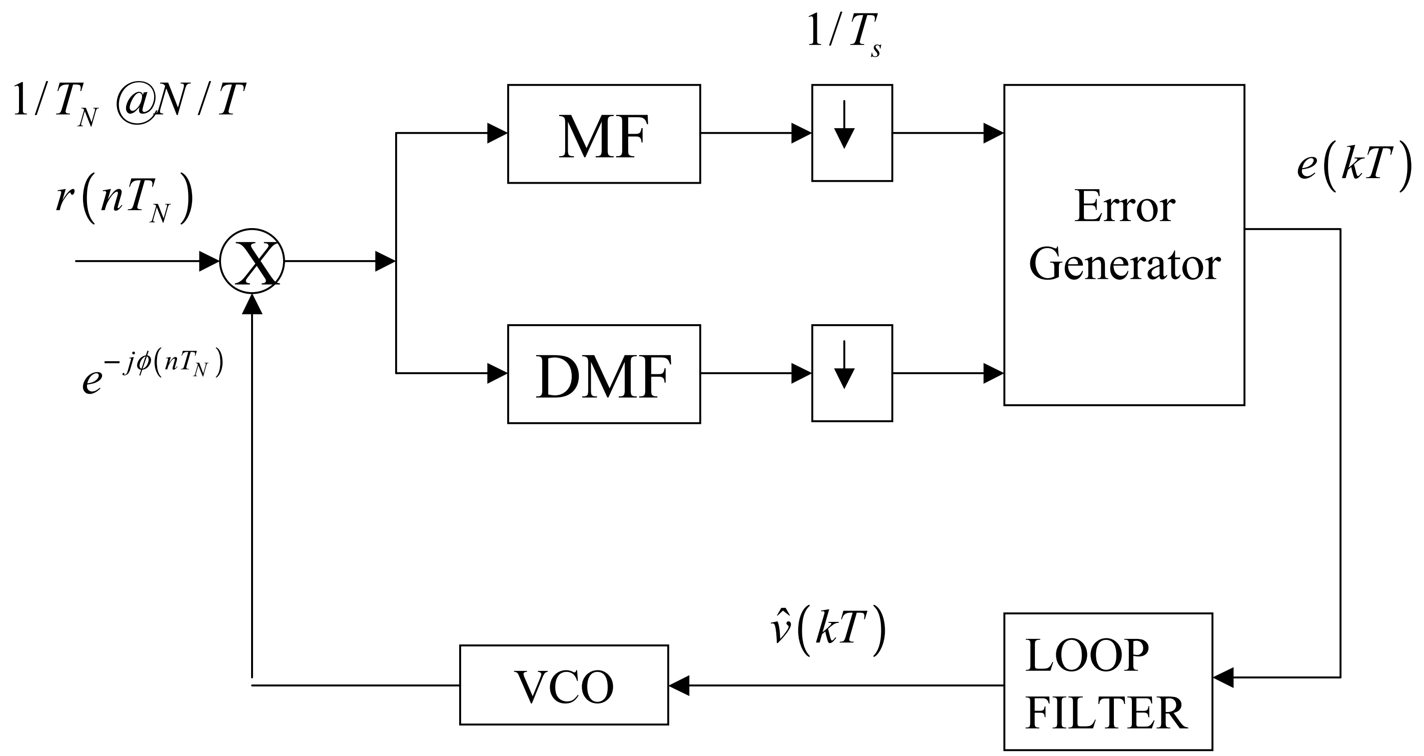
$$\frac{d\Lambda''(\mathbf{r}\mid\tilde{\vartheta})}{d\tilde{\vartheta}} = 2\int_0^{T_0} \operatorname{Im}\left\{ x(t) y^*(t) \right\} dt$$



derivative matched filter

$$H(f) = j\frac{dG^*(f)}{df}$$

$$\hat{\upsilon}\left[(k+1)T_s\right] = \hat{\upsilon}(kT_s) + \gamma u(kT_s) \qquad u(kT_s) = \sum_{i=k-N+1}^{k} \mathrm{Im}\left\{x(iT_s)\, y^*(iT_s)\right\}$$



$$\phi\left[(n+1)T_N\right] = \phi(nT_N) + 2\pi\hat{\upsilon}(kT)T/N \qquad \mathrm{mod}\, 2\pi$$

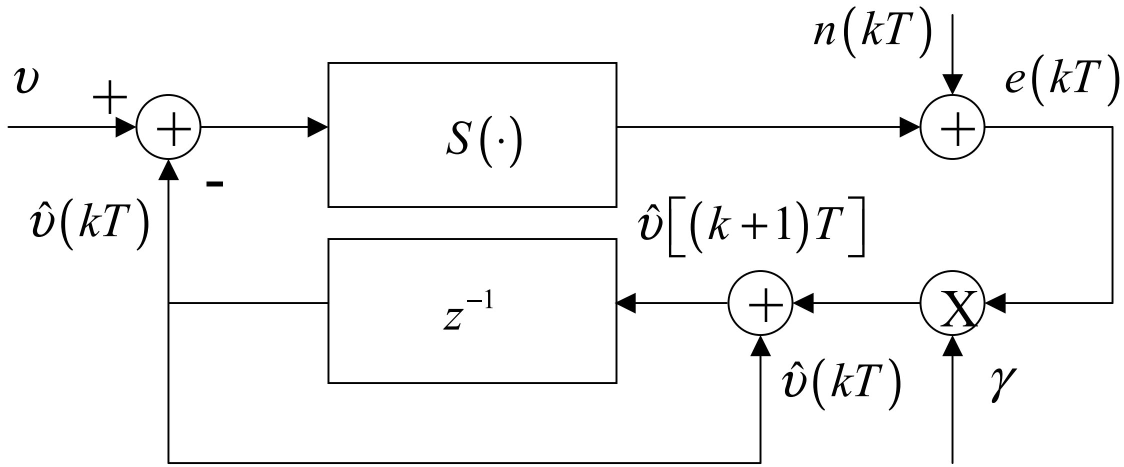
Frequency Acquisition

$$S\left(f_d\right) @ E\left\{e\left(kT\right) \mid \hat{\upsilon}\right\} \quad f_d @ \upsilon - \hat{\upsilon}$$

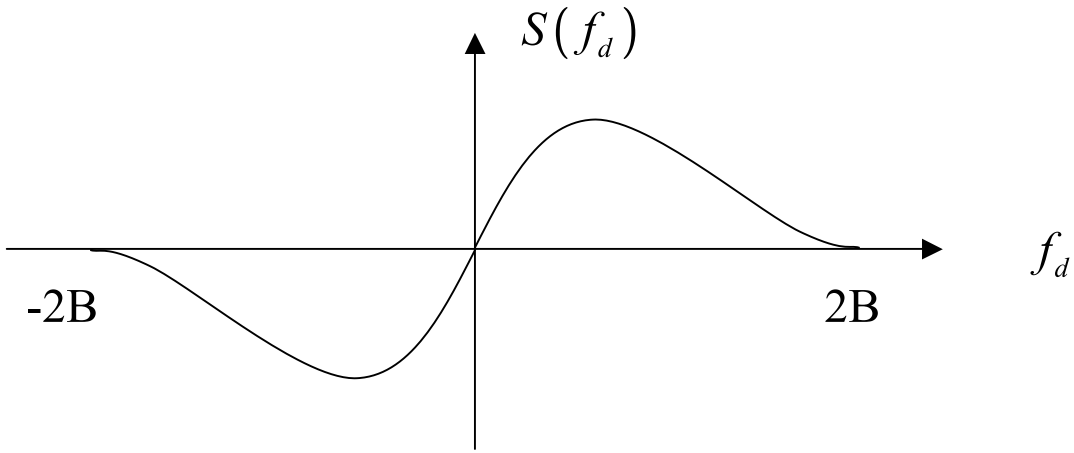
$$e\left(kT\right) = S\left[\upsilon - \hat{\upsilon}\left(kT\right)\right] + n\left(kT\right)$$

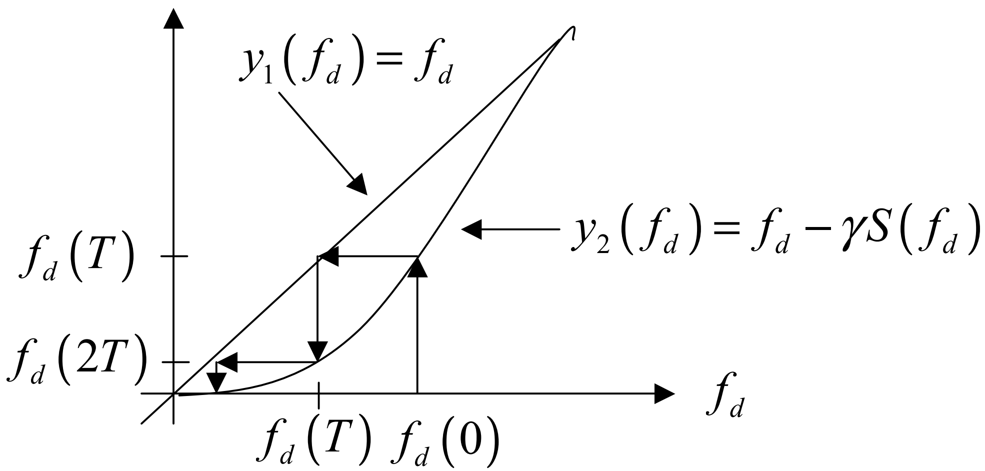
$$\hat{\upsilon}\left[\left(k+1\right)T\right] = \hat{\upsilon}\left(kT\right) + \gamma S\left[\upsilon - \hat{\upsilon}\left(kT\right)\right] + \gamma n\left(kT\right)$$

$$f_d\left[\left(k+1\right)T\right] = f_d\left(kT\right) + \gamma S\left[f_d\left(kT\right)\right]$$

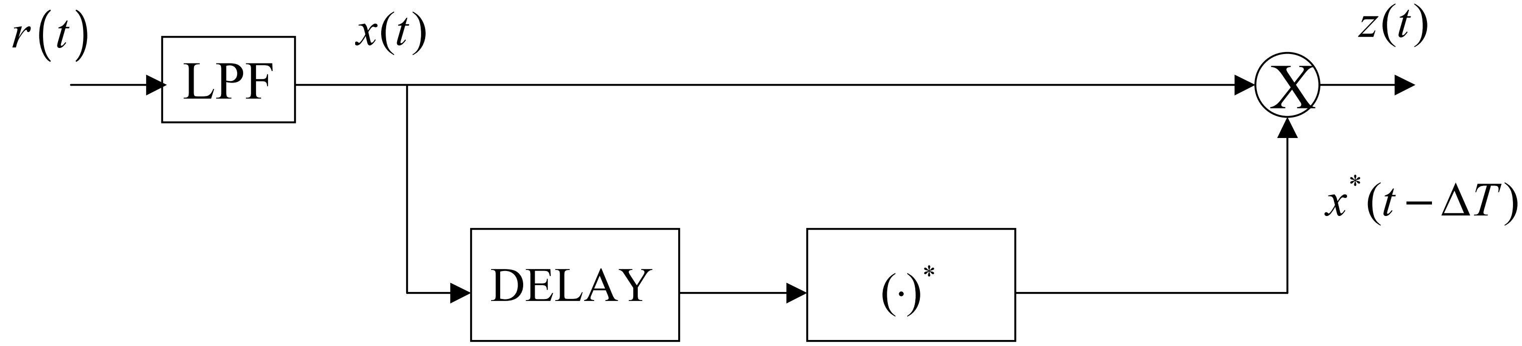


S-curve





OPEN-LOOP WITH NO TIMING INFORMATION



$$x(t) = e^{j(2\pi\upsilon t+\theta)} \sum_i c_i g(t - iT - \tau) + n(t)$$

$$z(t) = e^{j2\pi\upsilon\Delta T} \sum_i \sum_k c_i c_k^* g(t - iT - \tau) g(t - kT - \tau - \Delta T) +$$
$$s(t) n^*(t - \Delta T) + n(t) s^*(t - \Delta T) + n(t) n^*(t - \Delta T)$$

$$E\left[z(t)\right] = C e^{j2\pi\upsilon\Delta T} A(t - \tau) + R_n(\Delta T)$$

$$A(t) @ \sum_i g(t - iT) g(t - iT - \Delta T)$$ Periodic function of period T

$$z(t) = E\left[z(t)\right] + N(t)$$

$$\frac{1}{T_0}\int_0^{T_0} z(t)dt = CA_0 e^{j2\pi\upsilon\Delta T} + R_n(\Delta T) + X$$

DC component of A(t)

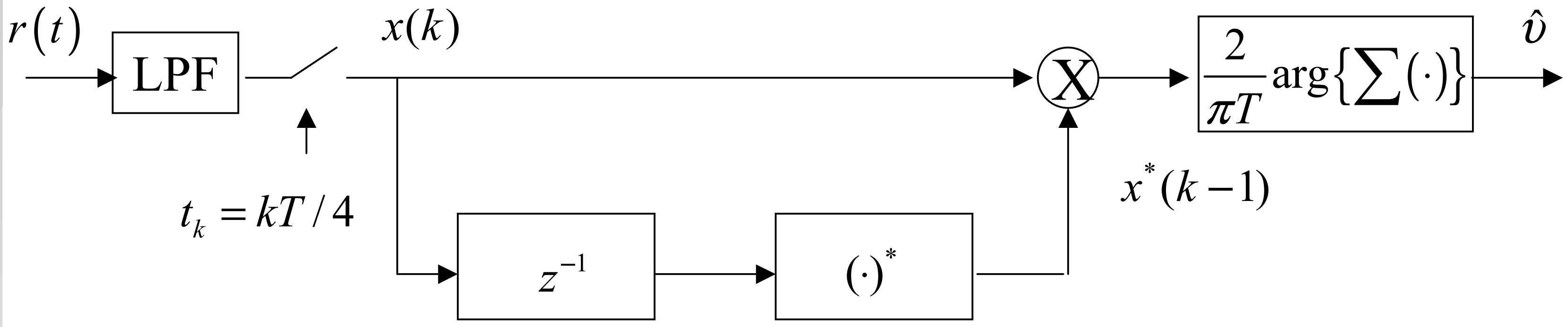
$$R_n(\Delta T) \approx 0$$

$$\Delta T = \frac{k}{2B_{LPF}}$$

$$X \; @ \frac{1}{T_0}\int_0^{T_0} N(t)dt$$

$$\hat{\upsilon} = \frac{1}{2\pi\Delta T}\arg\left\{\frac{1}{T_0}\int_0^{T_0} z(t)dt\right\}$$

Digital Implementation



$$\frac{1}{T_0}\int_0^{T_0} z(t)dt \approx \frac{T}{4}\sum_{k=0}^{4L_0-1} x(kT/4+t_0)x^*(kT/4+t_0-\Delta T)$$

$$\hat{\upsilon} = \frac{1}{2\pi\Delta T}\arg\left\{\sum_{k=0}^{4L_0-1} x(kT/4)x^*(kT/4-\Delta T)\right\}$$

$$\boxed{\hat{\upsilon} = \frac{2}{\pi T}\arg\left\{\sum_{k=0}^{4L_0-1} x(kT/4)x^*((k-1)T/4)\right\}} \qquad \Delta T = T/4$$