NON-DATA-AIDED DIGITAL SYNCHRONIZATION

Unconditional (or Stochastic) ML (UML)

Conditional (or Deterministic) ML (CML)

Minimum Conditioned Variance Compressed Maximum Likelihood (MCV-CML)

SIGNAL MODEL

$$r(t) = \sum_{n=-\infty}^{+\infty} c_n g(t - nT - \tau) e^{j(\theta_o + \omega t)} + w(t)$$

$$r(kT_s) = \sum_{n=-\infty}^{+\infty} c_n g(kT_s - nT - \tau) e^{j(\theta_o + \omega kT_s)} + w(kT_s)$$

$\mathbf{\Theta} = [\tau, v]^T$ Signal parameters of interest to be estimated

↓

$v = \frac{w}{2\pi} T$ Normalized frequency error

Nuisance parameters: carrier phase error and the transmitted symbols

Nss samples per symbol

Observation interval 2M+1 samples

If a single pulse is transmitted, the 2M+1 samples of the pulse are:

$$\mathbf{a}_n(\boldsymbol{\Theta}) = e^{-j\frac{2\pi}{N_{ss}}n_o v}\begin{bmatrix} g(-nT - MT_s - \tau)e^{-j\frac{2\pi}{N_{ss}}Mv} \\ g(-nT - (M-1)T_s - \tau)e^{-j\frac{2\pi}{N_{ss}}(M-1)v} \\ \vdots \\ g(-nT + (M-1)T_s - \tau)e^{j\frac{2\pi}{N_{ss}}(M-1)v} \\ g(-nT + MT_s - \tau)e^{j\frac{2\pi}{N_{ss}}Mv} \end{bmatrix}$$

n_0 constant that reflects the time origin

TDMA (Time Division Multiple Acces) : the transmitter sends a burst of 2k+1 symbols.

SCPC (Single Carrier Per Channel): the received data vector **r** will be affected by all the transmitted symbols, but, for a given pulse shape, only 2K+1 symbols will effectively be involved in the 2M+1 samples of the observation interval.

$$\mathbf{r} = \mathbf{A}(\boldsymbol{\Theta})\mathbf{x} + \mathbf{w}$$

$$\mathbf{A}(\boldsymbol{\Theta}) = \left[\mathbf{a}_{-K}(\boldsymbol{\Theta}) \quad \mathbf{a}_{-K+1}(\boldsymbol{\Theta}) \quad \cdots \quad \mathbf{a}_{K-1}(\boldsymbol{\Theta}) \quad \mathbf{a}_{K}(\boldsymbol{\Theta})\right]$$

$$\mathbf{x} = \left[x_{-K} \quad x_{-K+1} \quad \cdots \quad x_{K-1} \quad x_{K}\right]^T$$

$$x_n = c_n e^{j\theta_0} \qquad n = -K,\ldots,K$$

THE LINEAR SIGNAL MODEL INCLUDES:

- LINEAR MODULATIONS (*ASK*, *PSK*, *APK*, *QAM)*.
- SPREAD SPECTRUM MODULATIONS (*DS-CDMA*):
  - The pulse shape *g(t)* is the user signature, which includes the PN spreading sequence and the chip pulse.
- MULTI-CARRIER MODULATIONS (*OFDM*):
  - The signal model includes an arbitrary number of superimposed signals.
  - The shape pulses are complex.

$$x(t) = \sum_{l} \sum_{m=0}^{J-1} c_l^m \psi_m(t - lT) \longleftarrow \int_0^T \psi_k(t)\psi_i^*(t)dt = \delta_{ik}$$

NON-LINEAR MODULATIONS

- CONTINUOUS PHASE (Binary *CPM*): constant envelope and spectral efficiency.

$$r(t) = \sqrt{\frac{2E_s}{T}} e^{j\Phi(t;\boldsymbol{\alpha})} + w(t)$$

$$\Phi(t;\alpha) = 2\pi h \sum_n \alpha_n \int_{-\infty}^{t} q(t - nT)dt = \sum_n \alpha_n p(t - nT)$$

$$q(t) \neq 0 \quad for \quad 0 \leq t \leq LT$$

$$q(t) = q(LT - t)$$

- LAURENT EXPANSION:

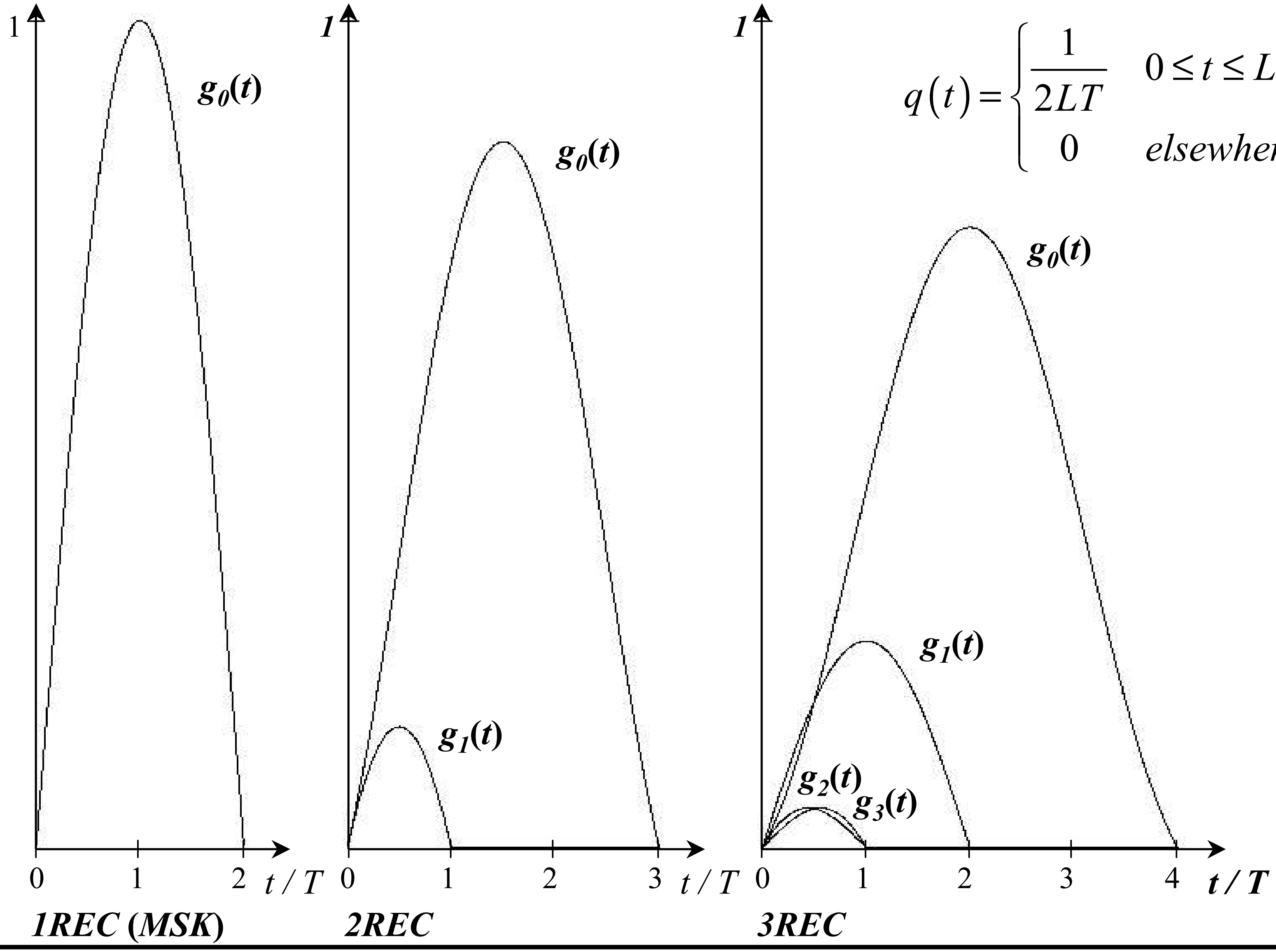
$$r(t) = \sum_{n=-\infty}^{+\infty} \sum_{m=0}^{J-1} c_{m,n} g_m (t - nT - \tau) e^{j(\theta_o + \omega t)} + w(t) \qquad J = 2^{L-1}$$

$$r(kT_s) = \sum_{n=-\infty}^{+\infty} \sum_{m=0}^{J-1} c_{m,n} g_m (kT_s - nT - \tau) e^{j(\theta_o + \omega kT_s)} + w(kT_s)$$

Uncorrelated Pseudosymbols: $E\left[c_{m,n} c^*_{m',n'}\right] = \delta_{m,m'} \delta_{n,n'}$

Asynchronous DS-CDMA $\left\{\tau_i\right\}_{0 \leq i \leq J-1}$

Pulses in LREC binary CPM modulations



$$q(t)=\begin{cases}\dfrac{1}{2LT} & 0\le t\le LT\\ 0 & elsewhere\end{cases}$$

If a single pulse is transmitted, the 2M+1 samples of the pulse are:

$$\mathbf{a}_{m,n}(\mathbf{\Theta}) = e^{-j\frac{2\pi}{N_{ss}}n_o v} \begin{bmatrix} g_m(-nT - MT_s - \tau)e^{-j\frac{2\pi}{N_{ss}}Mv} \\ g_m(-nT - (M-1)T_s - \tau)e^{-j\frac{2\pi}{N_{ss}}(M-1)v} \\ \vdots \\ g_m(-nT + (M-1)T_s - \tau)e^{j\frac{2\pi}{N_{ss}}(M-1)v} \\ g_m(-nT + MT_s - \tau)e^{j\frac{2\pi}{N_{ss}}Mv} \end{bmatrix}$$

$$m = 0,1,\ldots,J-1 \quad n = -K,\ldots,K$$

n_0 constant that reflects the time origin

$$\boxed{\mathbf{r} = \mathbf{A}(\boldsymbol{\Theta})\mathbf{x} + \mathbf{w}}$$

Superimposed Pulses — Multiplicative noise — AWGN

$$\mathbf{A}(\boldsymbol{\Theta}) = \left[\mathbf{A}_0(\boldsymbol{\Theta}) \quad \mathbf{A}_1(\boldsymbol{\Theta}) \quad \cdots \quad \mathbf{A}_{J-1}(\boldsymbol{\Theta})\right]$$

$$\mathbf{A}_m(\boldsymbol{\Theta}) = \left[\mathbf{a}_{m,-K}(\boldsymbol{\Theta}) \quad \mathbf{a}_{m,-K+1}(\boldsymbol{\Theta}) \quad \cdots \quad \mathbf{a}_{m,K-1}(\boldsymbol{\Theta}) \quad \mathbf{a}_{m,K}(\boldsymbol{\Theta})\right]$$

$$\mathbf{x} = \left[\mathbf{x}_0^T \quad \mathbf{x}_1^T \quad \cdots \quad \mathbf{x}_{J-1}^T\right]^T$$

$$\mathbf{x}_m = \left[x_{m,-K} \quad x_{m,-K+1} \quad \cdots \quad x_{m,K-1} \quad x_{m,K}\right]^T$$

$$x_{m,n} = c_{m,n} e^{j\theta_0}$$

$$n = -K, \ldots, K \qquad m = 0, 1, \ldots J-1$$

UNCONDITIONAL ML (UML)

$$\Lambda(\mathbf{r}/\mathbf{\Theta},\mathbf{x}) = C\exp(-\frac{1}{\sigma^2}q(\mathbf{r}/\mathbf{\Theta},\mathbf{x}))$$

$$q(\mathbf{r}/\mathbf{\Theta},\mathbf{x}) = \left\|\mathbf{r}-\mathbf{A}(\mathbf{\Theta})\mathbf{x}\right\|^2$$

The nuisance parameters in **x** are considered random.

Marginal Likelihood function of the wanted parameters: $\Lambda(\mathbf{r}/\mathbf{\Theta}) = E_{\mathbf{x}}\left[\Lambda(\mathbf{r}/\mathbf{\Theta},\mathbf{x})\right]$

Low SNR: $\Lambda(\mathbf{r}/\mathbf{\Theta},\mathbf{x}) \cong C - \frac{1}{\sigma^2}q(\mathbf{r}/\mathbf{\Theta},\mathbf{x}) + \frac{1}{2\sigma^4}q^2(\mathbf{r}/\mathbf{\Theta},\mathbf{x})$

$$\Lambda(\mathbf{r}/\mathbf{\Theta},\mathbf{x}) \cong C - \frac{1}{\sigma^2}\left(2\operatorname{Re}\left[\mathbf{x}^H\mathbf{A}^H\mathbf{r}\right] - \mathbf{x}^H\mathbf{A}^H\mathbf{A}\mathbf{x}\right) + \frac{1}{2\sigma^4}\left(2\operatorname{Re}\left[\mathbf{x}^H\mathbf{A}^H\mathbf{r}\right] - \mathbf{x}^H\mathbf{A}^H\mathbf{A}\mathbf{x}\right)^2$$

$\mathbf{A}^H\mathbf{A}$ does not depend on the parameteres

$$\Lambda(\mathbf{r}/\mathbf{\Theta},\mathbf{x}) \cong C - \frac{2}{\sigma^2}\operatorname{Re}\left[\mathbf{x}^H\mathbf{A}^H\mathbf{r}\right] + \frac{2}{\sigma^4}\operatorname{Re}^2\left[\mathbf{x}^H\mathbf{A}^H\mathbf{r}\right]$$

$$\Lambda(\mathbf{r}/\mathbf{\Theta}) \cong C + \frac{1}{\sigma^4}\mathbf{r}^H\mathbf{A}\mathbf{\Gamma}\mathbf{A}^H\mathbf{r}$$

$$\mathbf{\Gamma} = E\left[\mathbf{x}\mathbf{x}^H\right]$$

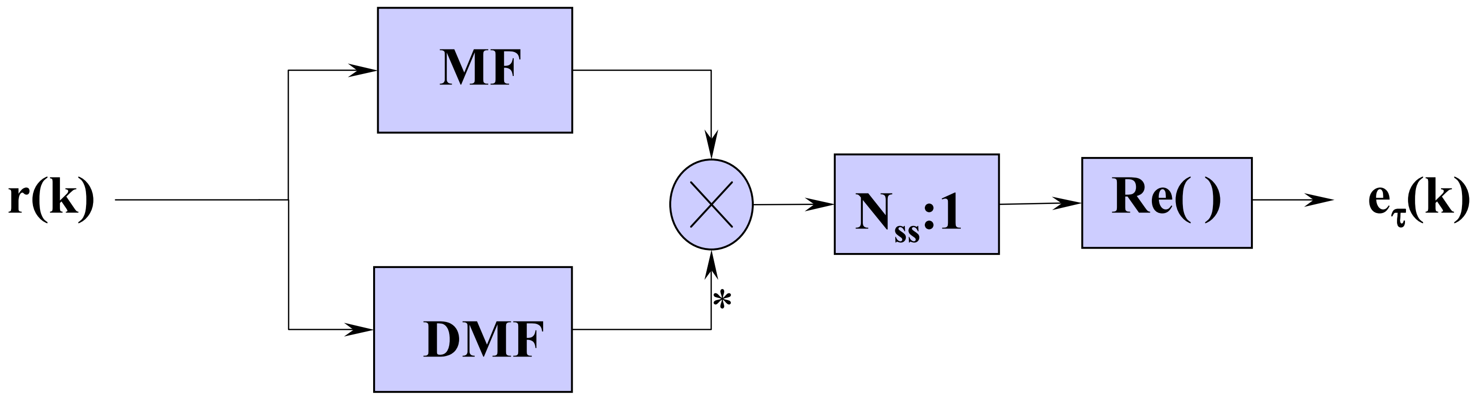
$$L_{UML}(\mathbf{r}/\mathbf{\Theta}) = \ln\left(\Lambda(\mathbf{r}/\mathbf{\Theta})\right) \approx \mathbf{r}^H\mathbf{A}\ \ \Gamma\ \ \mathbf{A}^H\mathbf{r} = \left\| \Gamma^{1/2}\ \ \underbrace{\mathbf{A}^H\mathbf{r}}_{\text{Matched filter output}} \right\|^2$$

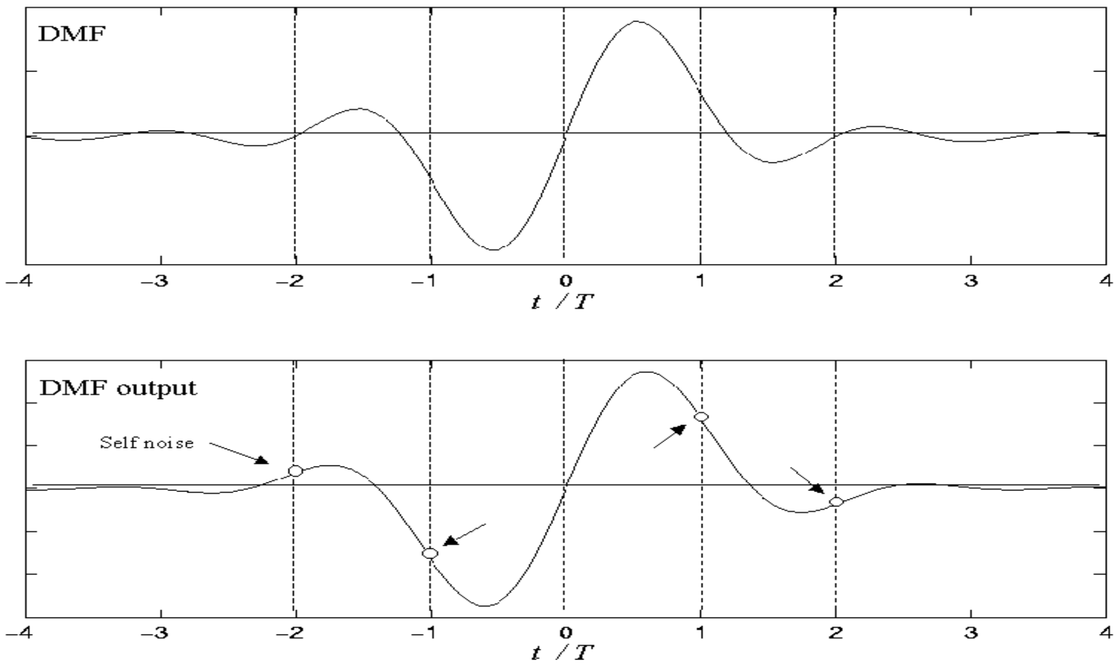
Matched filter output

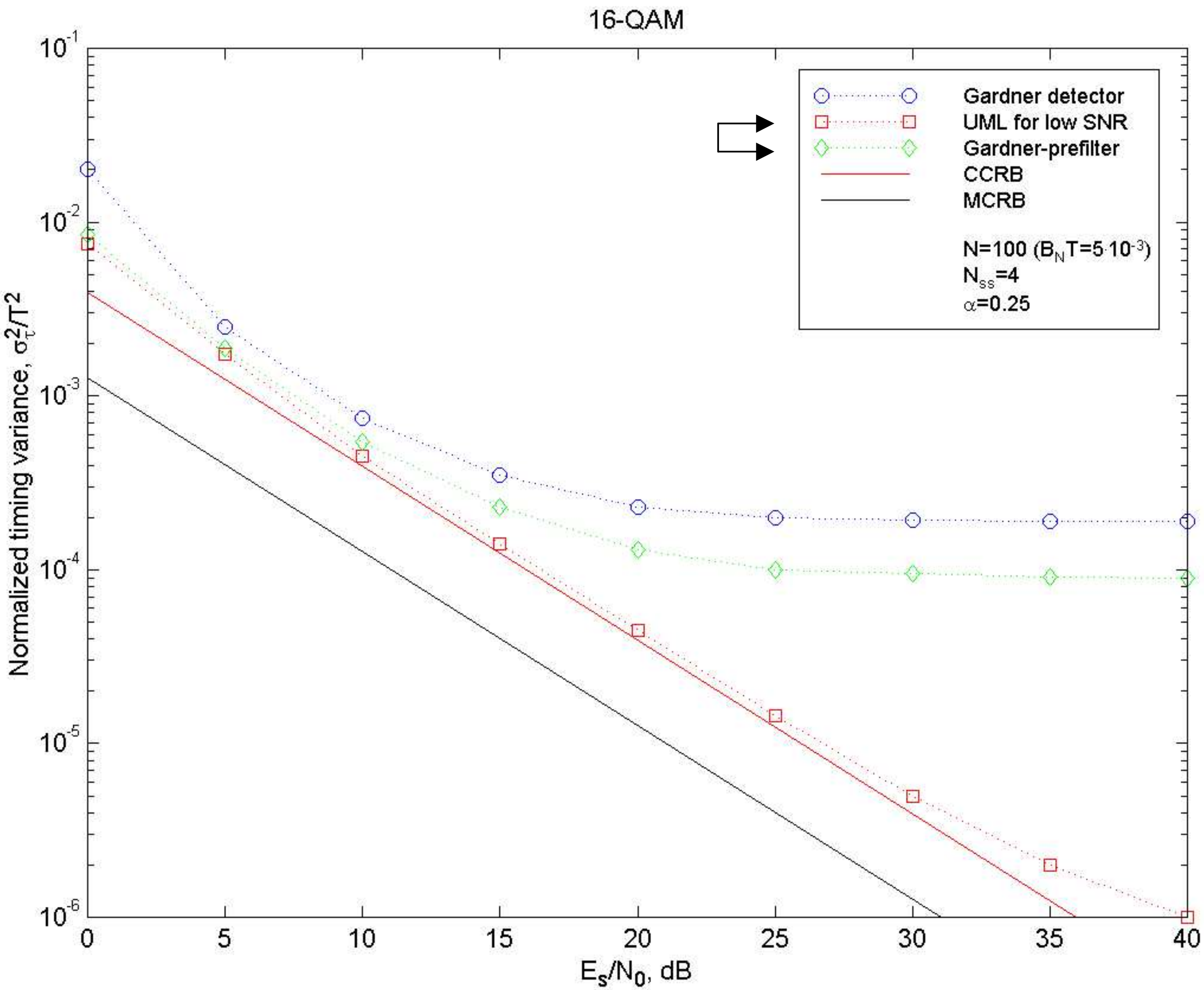
$$\frac{\partial}{\partial\Theta}L_{UML}(\mathbf{r}/\Theta) = \mathrm{Re}\left[\mathbf{r}^H\mathbf{D}\ \ \mathbf{\Gamma}\ \ \underbrace{\mathbf{A}^H\mathbf{r}}_{\text{Matched filter output}}\right]$$

Matched filter output

$$\mathbf{D}(\mathbf{\Theta}) = \frac{\partial}{\partial\mathbf{\Theta}}\mathbf{A}(\mathbf{\Theta})$$


r(k)
MF
DMF
*
N_ss:1
Re( )
e_τ(k)


DMF
–4
–3
–2
–1
0
1
2
3
4
t / T
DMF output
Self noise
–4
–3
–2
–1
0
1
2
3
4
t / T


16-QAM
Gardner detector
UML for low SNR
Gardner-prefilter
CCRB
MCRB
N=100 ($B_N T=5\cdot10^{-3}$)
N_{ss}=4
α=0.25
Normalized timing variance, σ_τ^2/T^2
E_s/N_0, dB
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
0
5
10
15
20
25
30
35
40

CONDITIONAL ML (CML)

The nuisance parameters in **x** are unknown deterministic variables.

$$\mathbf{r} = \mathbf{A}(\boldsymbol{\Theta})\mathbf{x} + \mathbf{w}$$

$$\hat{\mathbf{x}} = \mathbf{A}^{\#}(\boldsymbol{\Theta})\mathbf{r} = \mathbf{x} + \mathbf{A}^{\#}(\boldsymbol{\Theta})\mathbf{w}$$

ZERO FORCER EQUALIZER

$$\mathbf{A}^{\#} = \left(\mathbf{A}^{H}\mathbf{A}\right)^{-1}\mathbf{A}^{H} \longrightarrow$$

Symbol decorrelating matrix

$$L_{CML}(\mathbf{r}/\boldsymbol{\Theta}) = \ln\left(\Lambda(\mathbf{r}/\boldsymbol{\Theta},\hat{\mathbf{x}})\right) = \mathbf{r}^{H}\mathbf{A}\mathbf{A}^{\#}\mathbf{r}$$

$$\hat{\boldsymbol{\Theta}}_{CML} \rightarrow \max_{\boldsymbol{\Theta}} \mathbf{r}^{H}\mathbf{A}\mathbf{A}^{\#}\mathbf{r} \longrightarrow$$

Insensitive to the symbol autocorrelation matrix

$$\hat{\boldsymbol{\Theta}}_{CML} \rightarrow \min_{\boldsymbol{\Theta}} Tr\left[\mathbf{P}_{\mathbf{A}}^{\perp}\mathbf{r}\mathbf{r}^{H}\right]$$

$$\mathbf{P}_{\mathbf{A}}^{\perp} = \mathbf{I} - \mathbf{A}\mathbf{A}^{\#} \longrightarrow$$

Projection matrix onto the orthogonal signal subspace

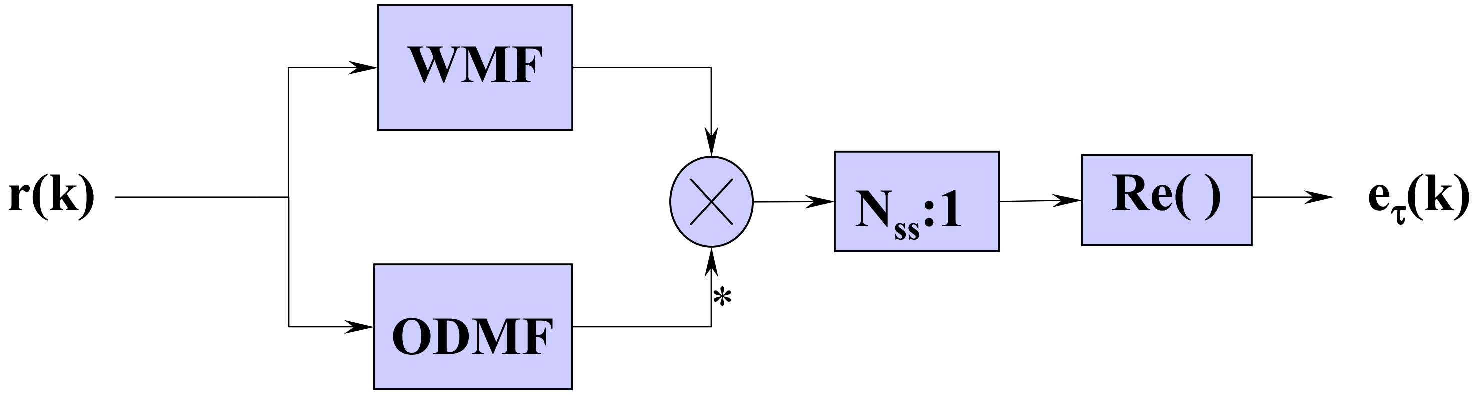
$$\frac{\partial}{\partial\Theta}L_{CML}(\mathbf{r}/\Theta)=\mathrm{Re}\left[\underbrace{\mathbf{r}^{H}\mathbf{P}_{\mathbf{A}}^{\perp}\mathbf{D}}\;\underbrace{\mathbf{A}^{\#}\mathbf{r}}\right]$$

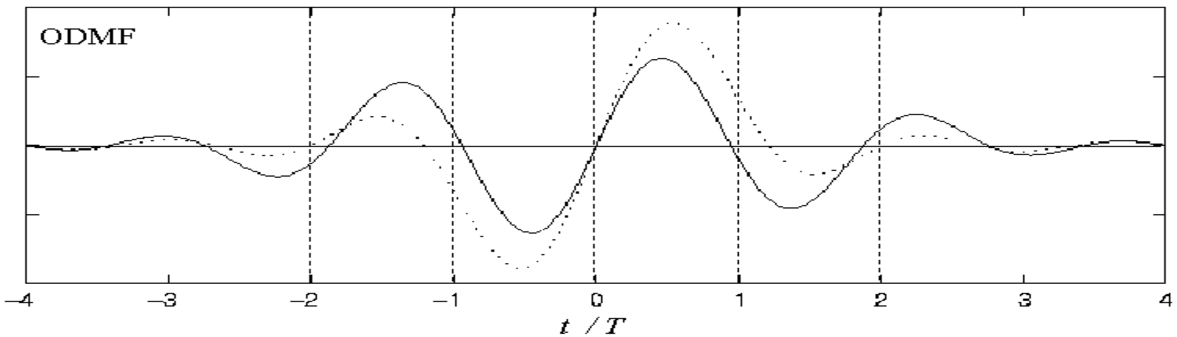
Orthogonal Derivative Matched Filter (ODMF) output → No self-noise

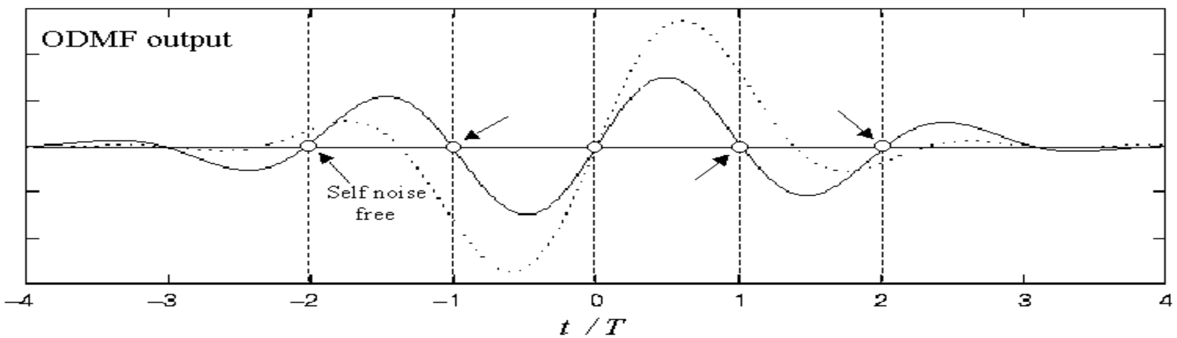
in the noiseless case:

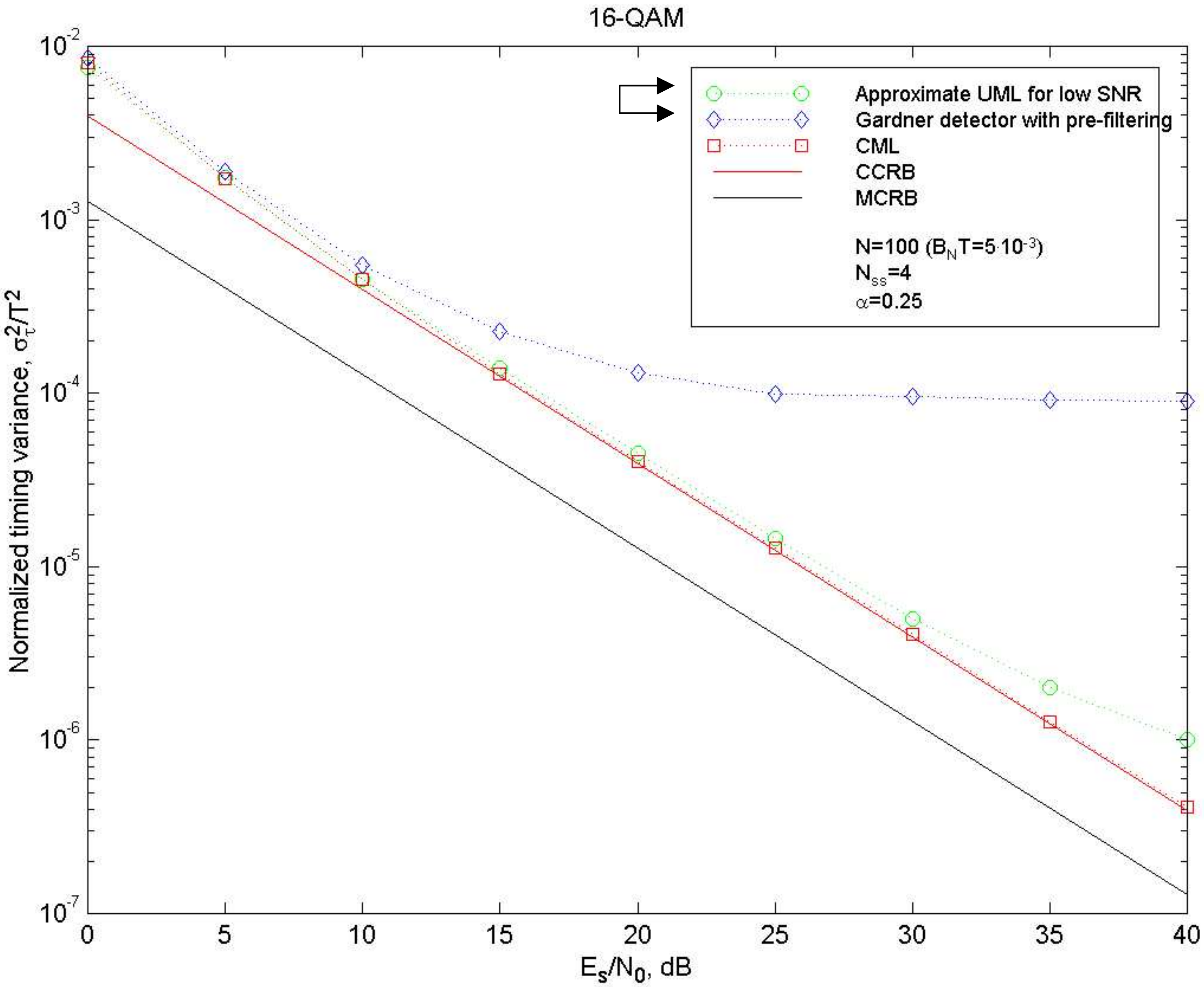
$$\mathbf{r}^{H}\mathbf{P}_{\mathbf{A}}^{\perp}\mathbf{D}\Big|_{\hat{\tau}=\tau,\hat{\nu}=\nu}=0$$

Decorrelation of the symbols → Increase of the noise

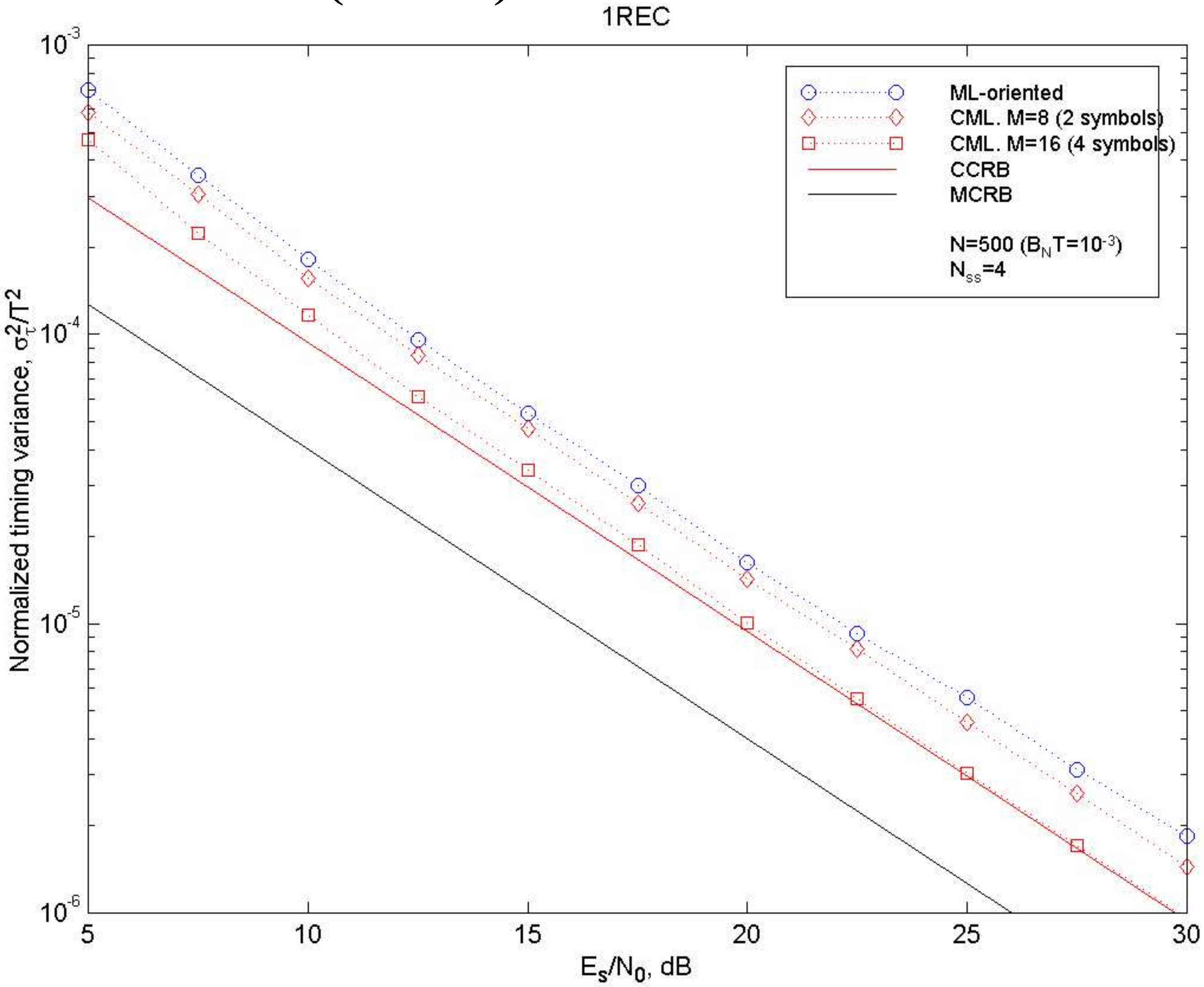

WMF
r(k)
ODMF
*
N_ss:1
Re( )
e_τ(k)


ODMF
t / T


ODMF output
Self noise free
t / T


16-QAM
Approximate UML for low SNR
Gardner detector with pre-filtering
CML
CCRB
MCRB
N=100 (B_N T=5·10^-3)
N_ss=4
α=0.25
Normalized timing variance, σ_τ^2/T^2
E_s/N_0, dB
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
0
5
10
15
20
25
30
35
40

MSK (1REC)


1REC
ML-oriented
CML. M=8 (2 symbols)
CML. M=16 (4 symbols)
CCRB
MCRB
N=500 ($B_N T=10^{-3}$)
N_{ss}=4
Normalized timing variance, σ_τ^2/T^2
E_s/N_0, dB
10^{-3}
10^{-4}
10^{-5}
10^{-6}
5
10
15
20
25
30

MINIMUM CONDITIONED VARIANCE COMPRESSED MAXIMUM LIKELIHOOD (MCV-CML)

The wanted parameter vector is defined such that the received data vector is best fitted, under a minimum square-error sense, to the regenerated data from the best linear estimate of the nuisance parameter vector:

$$\hat{\mathbf{x}} = E[\mathbf{x}/\mathbf{r}] = \mathbf{A}^H \left(\mathbf{A}\mathbf{A}^H + \sigma^2 \mathbf{I}\right)^{-1} \mathbf{r} = \mathbf{C}\mathbf{r}$$

$$L_{MCV}(\mathbf{r}/\boldsymbol{\Theta}) = \left\|\mathbf{r} - \mathbf{A}\hat{\mathbf{x}}\right\|^2 = \left\|\mathbf{r} - \mathbf{A}\mathbf{C}\mathbf{r}\right\|^2 = \left\|(\mathbf{I} - \mathbf{A}\mathbf{C})\mathbf{r}\right\|^2$$

$$\frac{\partial}{\partial \Theta} L_{MCV}(\mathbf{r}/\Theta) = -2\,\mathrm{Re}\left[\mathbf{r}^H (\mathbf{I} - \mathbf{A}\mathbf{C})\left(\mathbf{D}\mathbf{C} + \mathbf{A}\frac{\partial}{\partial \Theta}\mathbf{C}\right)\mathbf{r}\right]$$

$$\left(\mathbf{I}-\mathbf{AC}\right)\mathbf{DC}+\left(\mathbf{I}-\mathbf{AC}\right)\mathbf{A}\frac{\partial}{\partial\Theta}\mathbf{C}$$

Low SNR: **C** corresponds to the matched filter of the transmitted pulses: $\mathbf{C}=\sigma^{-2}\mathbf{A}^{H}$

$$\left(\mathbf{I}-\mathbf{AC}\right)\mathbf{DC}+\left(\mathbf{I}-\mathbf{AC}\right)\mathbf{A}\frac{\partial}{\partial\Theta}\mathbf{C}\cong\sigma^{-2}\left(\mathbf{DA}^{H}+\mathbf{AD}^{H}\right)$$

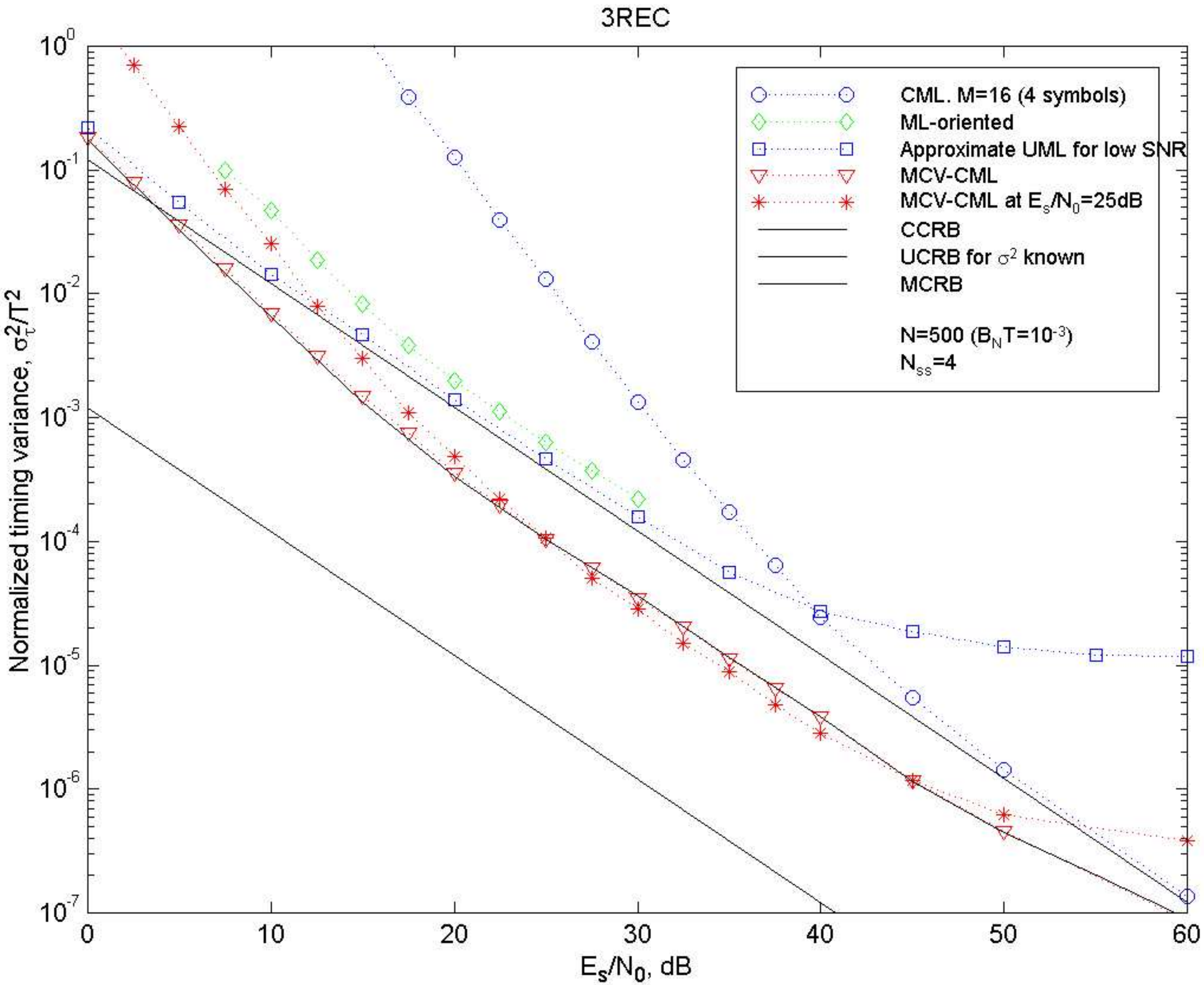
$$\frac{\partial}{\partial\Theta}L_{MCV}(\mathbf{r}/\Theta)=-\frac{2}{\sigma^{2}}\mathrm{Re}\left[\mathbf{r}^{H}\left[\left(\mathbf{DA}^{H}+\mathbf{AD}^{H}\right)\right]\mathbf{r}\right]=-\frac{4}{\sigma^{2}}\mathrm{Re}\left[\mathbf{r}^{H}\mathbf{DA}^{H}\mathbf{r}\right]\longrightarrow$$ Low-SNR UML

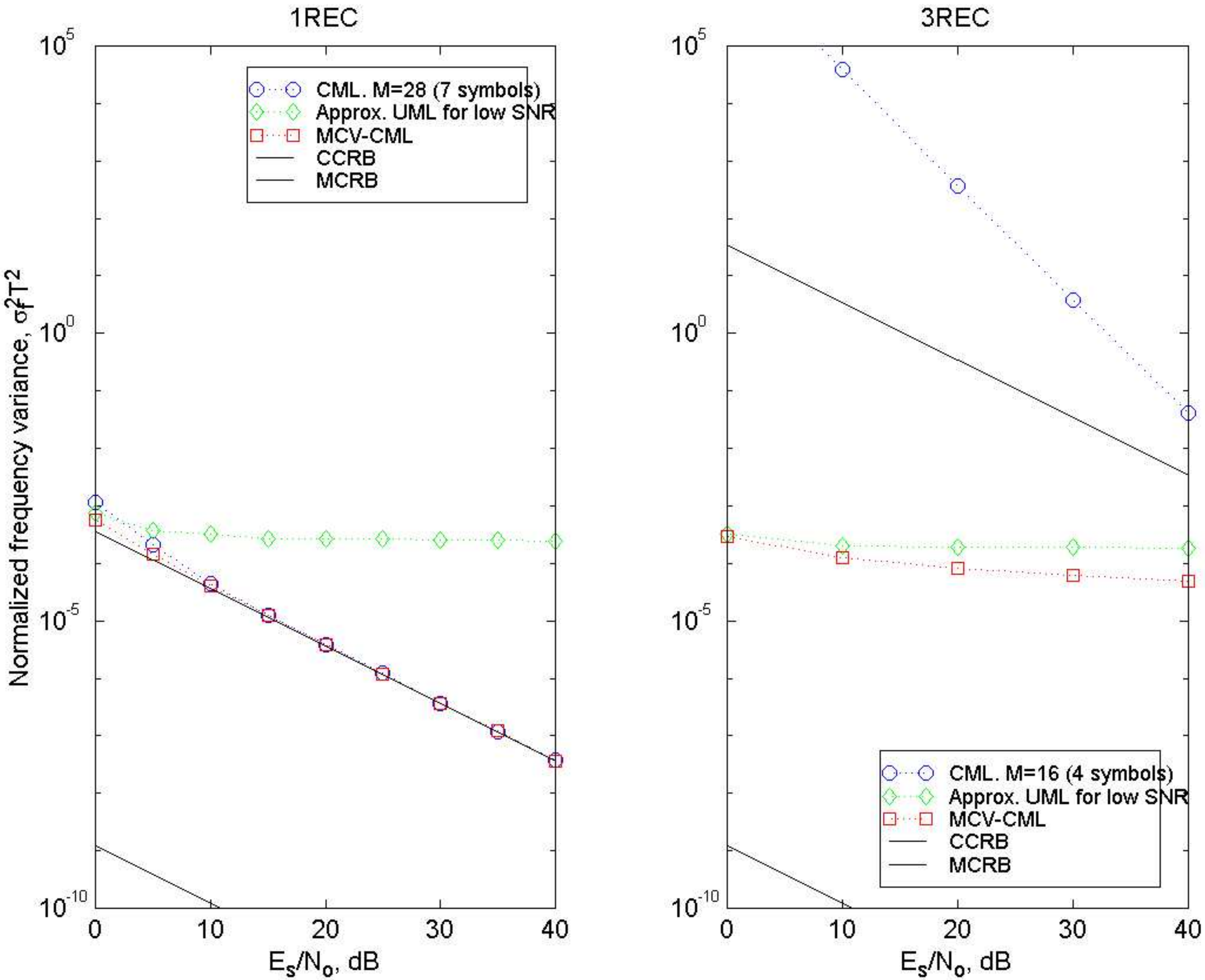
High SNR: **C** pseudo-inverse of **A:** $\mathbf{C}=\mathbf{A}^{\#}$

$$\left(\mathbf{I}-\mathbf{AC}\right)\mathbf{A}\frac{\partial}{\partial\Theta}\mathbf{C}\cong\left(\mathbf{I}-\mathbf{AA}^{\#}\right)\mathbf{A}\frac{\partial}{\partial\Theta}\mathbf{A}^{\#}=\mathbf{P}_{\mathbf{A}}^{\perp}\mathbf{A}\frac{\partial}{\partial\Theta}\mathbf{A}^{\#}\equiv 0$$

$$\frac{\partial}{\partial\Theta}L_{MCV}(\mathbf{r}/\Theta)=-2\,\mathrm{Re}\left[\mathbf{r}^{H}\mathbf{P}_{\mathbf{A}}^{\perp}\mathbf{D}\;\;\mathbf{A}^{\#}\mathbf{r}\right]\longrightarrow$$ CML

$$\boxed{\frac{\partial}{\partial\Theta}L_{MCV}(\mathbf{r}/\Theta)\approx\mathrm{Re}\left[\mathbf{r}^{H}\left(\mathbf{I}-\mathbf{AC}\right)\mathbf{D}\;\;\mathbf{C}^{H}\mathbf{r}\right]}$$


3REC
CML. M=16 (4 symbols)
ML-oriented
Approximate UML for low SNR
MCV-CML
MCV-CML at Es/N0=25dB
CCRB
UCRB for σ2 known
MCRB
N=500 (BNT=10-3)
Nss=4
Normalized timing variance, στ2/T2
Es/N0, dB
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
0
10
20
30
40
50
60


1REC
3REC
Normalized frequency variance, $\sigma_f^2T^2$
10^5
10^0
10^{-5}
10^{-10}
0
10
20
30
40
E_s/N_o, dB
CML. M=28 (7 symbols)
Approx. UML for low SNR
MCV-CML
CCRB
MCRB
CML. M=16 (4 symbols)

PERFORMANCE LIMITS

$$CRB(\lambda) = \frac{1}{E_{\mathbf{r}}\left\{\left|\frac{\partial}{\partial \lambda} \ln E_{\mathbf{x}} \Lambda(\mathbf{r}/\boldsymbol{\Theta}, \mathbf{x})\right|^2\right\}}$$

$$MCRB(\lambda) = \frac{1}{E_{\mathbf{r},\mathbf{x}}\left\{\left|\frac{\partial}{\partial \lambda} \ln \Lambda(\mathbf{r}/\boldsymbol{\Theta}, \mathbf{x})\right|^2\right\}} = \frac{\sigma^2}{2Tr\left(\mathbf{D}_{\lambda}^{H}\mathbf{D}_{\lambda}\boldsymbol{\Gamma}\right)}$$

$$CCRB(\lambda) = \frac{1}{E_{\mathbf{r}}\left\{\left|\frac{\partial}{\partial \lambda} \ln \Lambda(\mathbf{r}/\boldsymbol{\Theta}, \hat{\mathbf{x}})\right|^2\right\}} = \frac{\sigma^2}{2\mathbf{x}^{H}\mathbf{D}_{\lambda}^{H}\mathbf{P}_{\mathbf{A}}^{\perp}\mathbf{D}_{\lambda}\mathbf{x}}$$