Arnoldi Stability Analysis of Modulated Tollmien-Schlichting Waves in Shear Flows

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The transition of strictly two-dimensional Poiseuille flow from one-frequency amplitude-modulated states to chaotic behaviour is studied through full numerical simulation of spatially periodic channels with large longitudinal aspect ratios. First, time evolution on states at different Reynolds numbers is performed. Then, linear stability techniques, namely Poincaré maps combined with Arnoldi Iteration methods, are used to attain more exact results.

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I. INTRODUCTION

Direct application of Newton's second law to fluid motion, together with the assumption that the stress in fluids is the sum of a diffusion viscous term, proportional to the gradient of velocity, and a pressure term, gives raise to the Navier-Stokes equations. This equation can be written in terms of the velocity field $\boldsymbol{u}$ as

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u}$$

where ν is the kinematic viscosity of the fluid. This equation accurately describes the motion of viscous fluid substances. Few analytical solutions can be found to this equation, so most of the work must be done computationally.

In this paper we consider two-dimensional Poiseuille flow, that is, a fluid confined between two rigid fixed plates and driven by an externally imposed pressure gradient.

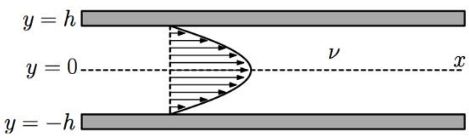


FIG. 1. Two-dimensional Poiseuille flow. The basic flow solution of the N-S equation is represented.

Analytically, if the velocity u_z perpendicular to the paper is set to zero, a solution for the velocity in the x-y plane can be found. It yields u_y=0 and

$$u_x = -\frac{1}{2\mu}\frac{dp}{dx}y(h-y)$$

where μ is the dynamic velocity and where x is the streamwise direction and y the coordinate perpendicular to the walls. No-slip boundary conditions in the walls have been imposed and infinitely large x and z directions have been considered.

This parabolic base flow is always a solution of the problem, but not always an stable one. Linear viscous calculations predict that the flow becomes unstable at a critical Reynolds number of 5780. Nonlinear calculations give a critical *Re* of 2510, which agrees better with the observations of transition to turbulence.[**1**] The disturbances that cause instability in these flows are called Tollmien-Schlichting waves. This structures vary as a function of the Reynolds number, appearing first as stable constant-amplitude waves, presenting a one-frequency amplitude modulation for higher *Re* and finally adding another frequency to its amplitude modulation just before transitioning to chaos and turbulence.

A computational method for finding the exact *Re* number at which a second frequency in the Tollmien-Schlichting waves amplitude modulation appears is presented here. First, temporal evolution using Navier-Stokes equation is used to find stable solutions. Then, an stability analysis performed using both Poincaré maps and Arnoldy's method to find specific eigenvalues is used to certify the obtained results.

II. NUMERICAL APPROACH FOR SOLVING N-S EQUATIONS

The first calculations involve temporal evolution to find stable states for different Reynolds numbers. This process requires solving Navier-Stokes equations multiple times at – ideally- every point of the domain. This way, with proper boundary conditions, the motion of the fluid can be properly modelled. Thus, a fast computational method for solving these equations with the conditions and domain we are about to use needs to be applied.

Navier-Stokes equations are, indeed, a set of two or three coupled equations, one for each dimension. Henceforth we adopt, instead, the streamfunction formulation of Drissi et al. (1999) that transforms two second order coupled PDEs to one fourth order PDE concerning an scalar called streamfunction, $\Psi(x, y; t)$, that is related to the two-dimensional velocity field through the expressions $u_x = \partial_y \psi$ and $u_y = -\partial_x \psi$.[2] It can be proved that the Navier-Stokes equations in two dimensions can be written under this formalism as

$$\partial_t \nabla^2 \psi + (\partial_y \psi)\partial_x(\nabla^2 \psi) - (\partial_x \psi)\partial_y(\nabla^2 \psi) = \frac{1}{Re}\nabla^4 \psi$$

Boundary conditions need also to be transformed to this formulation. In our case, the no-slip boundary conditions in the walls can be written as

$$\partial_y \psi(x, \pm 1; t) = \partial_x \psi(x, \pm 1; t) = 0$$

Note that, for simplicity and without losing any generality, we have rescaled the y direction so as to have the walls placed at $y=\pm 1$.

The original problem consists of studying a flow that is infinite in the x direction. This cannot be attained neither experimentally nor computationally. In our case, streamwise-periodic boundary conditions were adopted. Finally, a constant mean axial velocity condition was also imposed.

Full knowledge of the streamfunction in the full domain is enough to represent the fluid state of motion. Solutions of the problem are approximated using a Fourier-Legendre expansion:

$$\psi_{LM}(x, y, t) = \sum_{l=-L}^{L} \sum_{m=0}^{M} a_{lm}(t) e^{ilkx} \phi_m(y)$$

where k is the minimum streamwise wavenumber and $\Phi_m(y)=(1-y^2)^2 L_m(y)$, with $L_m(y)$ the m-th Legendre polynomial. [2] In this paper, the number of Fourier modes, L, has been set to 35 and M=50.

Note that k can be written as $2\pi/\Lambda$, where Λ is the length of the periodic domain in non-dimensional units –take into account that between walls, there is a distance of 2. Ideally, Λ should be infinite. A finite value has to be picked, and this election affects strongly the obtained results. This is, in fact, a major drawback, as, if the method was robust, the election Λ, if big enough, would not influence much the results. In fluid mechanics, though, lots of approximations need to be made. In this case, the elected value was $\Lambda=26$ and $k=0.24$.

The complete set of a_{lm} at a given instant of time fully characterizes, under the given approximations, the state of the fluid motion in the specified domain. That is, each state of motion can be regarded as a point in an LxM-dimensional space.

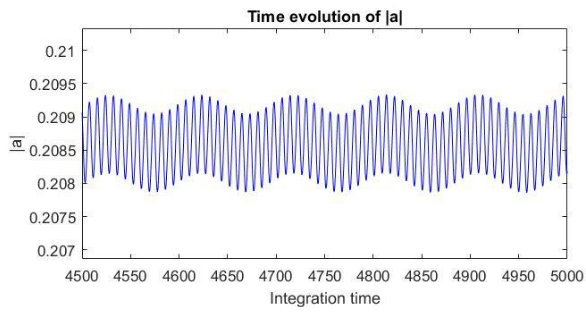
III. METHODOLOGY

The calculations started with a converged state at $Re=5000$ that presented waves with one-frequency amplitude modulation. This state, as it is stable, can be found by temporal evolution, that is, by discretizing time and solving the Navier-Stokes equations many times per unit time. However, the initial state need to be in the basin of attraction of this solution, so finding such a state might not be easy. In any case, we did not perform this calculation, as the converged state that served as the starting point for our calculations was given to us.

Starting with this solution, represented by the set of a_{lm}, we started increasing Re with increments of 50 units. For each increment of Re, the Navier–Stoques equations were integrated during 40 000 units of time -100 integrations/unit time-, using as initial condition the converged stable state obtained by time evolution in the preceding Re. This way, we always started in the basin of attraction of the stable solution for the new Re. Representing the norm of the amplitude of the wave as a function of time and its FFT, we could determine if one or two frequencies were present in its modulation.

We concluded that for Re=5100, only one frequency is present but, for Re=5150, there are already two frequencies (as seen in FIG. 2a and b), so the critical Re must be somewhere between these two values.

a.



b.

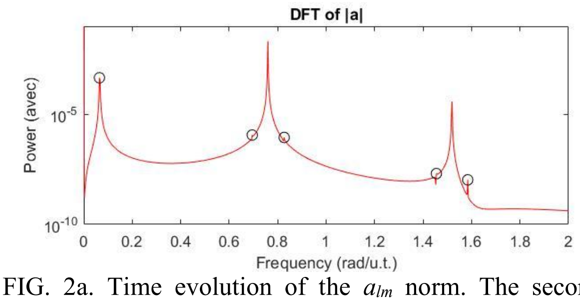


FIG. 2a. Time evolution of the a_{lm} norm. The second modulation can be clearly seen. 2b. DFT of the norm of the coefficients a_{lm}. The second harmonic is also visible. The black circles show the second frequency.

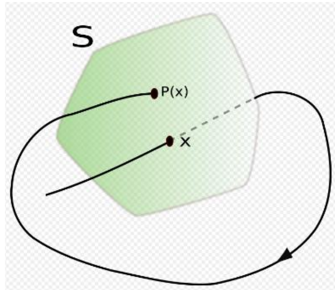
At this point, two different methods based on Poincaré maps where applied to find the exact Re at which the one-frequency amplitude-modulated solution becomes unstable and the fluid motion becomes a two-frequency amplitude-modulated state.

If we consider the LxM-dimensional space of a_{lm}, a one-frequency amplitude-modulated state can be represented as a periodic orbit, where the system goes through many different states in a periodic manner. If a second frequency is to be introduced, it can be thought of as a toroid. For *Re* smaller than the critical *Re* that is being searched for, the single periodic orbit is stable. For a greater Re, this orbit becomes unstable but still exists. It does so inside a toroid that has become stable.

A Poincaré map is the intersection of a periodic orbit in the state space of a continuous dynamical system with a certain lower-dimensional subspace, called the Poincaré section. More precisely, it is a map defined between the points in the section from which orbits depart and the points where these orbits return after a full period. The perpendicularity of the Poincaré section means that periodic orbits starting on the subspace flow through it and not parallel to it. [**3**]

In order to find the exact *Re* at which the one-frequency modulated solution becomes unstable, the following process was fulfilled: first, the time-evolution obtained state at Re=5100 was converged using a Newton's method on the equation P(***x***)-***x***=0, yielding a solution $\boldsymbol{x_o}$, where $\boldsymbol{x_0}$ is the LxM-dimensional vector defining the motion state and P(***x***) the Poincaré map. Once the exact one-frequency modulated solution $\boldsymbol{x_0}$ was obtained, the biggest eigenvalues of P(***x***) were computed using Arnoldi's method. The reason for that is that, if any eigenvalue of P(***x***) had a module greater than 1, then the orbit would be unstable in the direction of the eigenvector associated with that eigenvalue, so the one-frequency modulated solution would be unstable and a second frequency would appear. Finally, random perturbations were induced in the periodic orbit and their successive intersections with the Poincaré section where recorded, analysing the results and checking that they were congruent with the previous observations.

Obviously, for Re=5100 the stability analysis determined that the orbit was stable. From here on, *Re* was incremented in steps of 3 units up to 5148 and the same analysis was performed every time, until the orbit was found to become unstable. That would be the critical *Re* we were looking for.



I. FIG. 3. Illustration of a Poincaré map. The orbit would be periodic if P(***x***)=***x***.

II. RESULTS

Both methods, calculating the eigenvalues with Arnoldi's method and studying small perturbations with the Poincaré map, gave the same result. The solution with only one modulation becomes unstable at Re≃5125, giving rise to a double modulated solution. As in the direction perpendicular both to the walls and the propagation direction, there is always an eigenvalue equal to one which doesn't have to be taken into account.

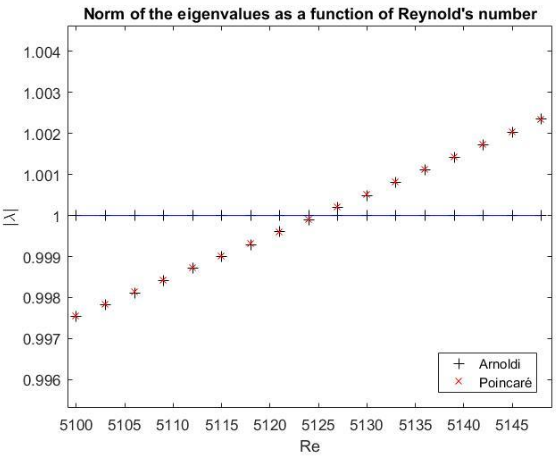


FIG. 4. Norm of the biggest eigenvalues obtained with each of the methods.

The time evolution of the perturbations on the Poincaré map showed clearly when the one modulation solution was stable and when not. Even though the plot is the norm of the perturbation, due to it being a complex pair, the graph is not a line in semi logarithmic axis but it has oscillations. The norm of the biggest eigenvalue however, is easily obtained with a linear regression because its logarithm is the slope.

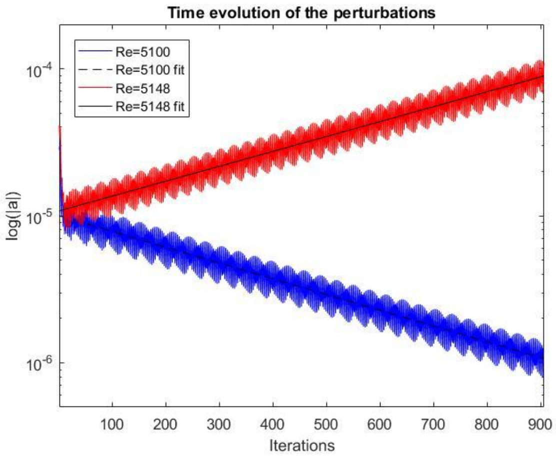


FIG. 5. Time evolution of the perturbations and their fits. As it can be seen, if the perturbation diverges, the slope is bigger than one which means that the eigenvalue has already crossed the unitary circle.

Arnoldi's method allows to know not only the eigenvalue norm but also its phase, and hence, the eigenvalues can be plotted in the complex plane to determine the kind of transition. In the case of study, the eigenvalues whose norm became bigger than one where a conjugated pair, giving rise to the second modulation as seen in the DFT.

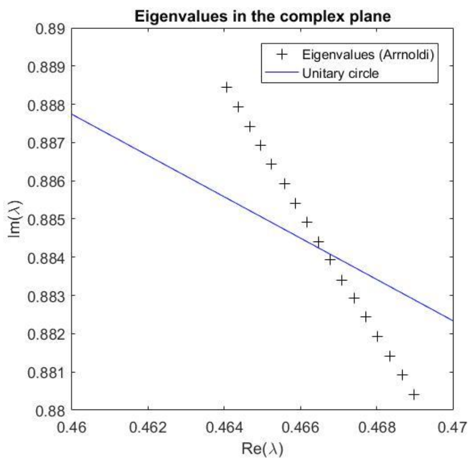


FIG. 6. Representation of the eigenvalues that undergo the transition to instability between Re=5100 and Re=5148 in the complex plane.

III. CONCLUSIONS

We have adopted a streamfunction formulation for two-dimensional flow and a set of numerical methods in order to solve and study with detail the instability transition of the modulated TSW for k=0.24.

We conducted the study using three different approaches: temporal evolution following the N-S equation, Arnoldi and perturbations in the Poincaré map. All three approaches gave the same result. The modulated TSW, which is the stable solution for Reynolds around 5000, undergoes a transition to instability at Re≃5125 becoming a TSW with two modulations.

This transition to instability is due to the crossing of a complex pair of eigenvalues. However, there are other eigenvalues near crossing.

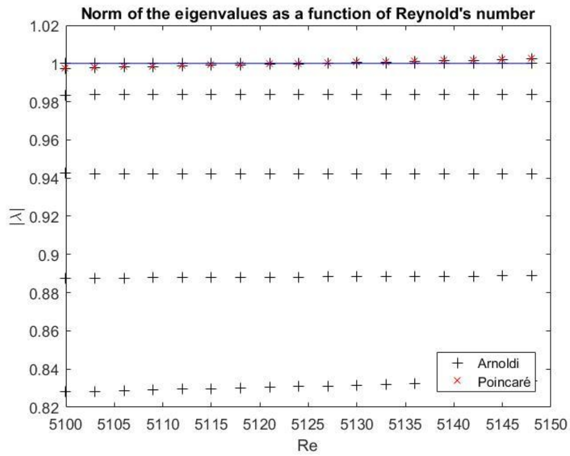


FIG. 7. All the biggest eigenvalues found with Arnoldi's method increase with Reynolds.

This result is important to study the transition to chaotic flux that fluids experiment when the Reynolds number increases. In this case, the most probable mechanism for chaotic transition is the cascade of complex eigenvalues which would progressively increase the modulations of the TSW until the flux becomes caothic.

1. ACKNOWLEDGEMENTS

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1. [**1**] P.K. Kundu et al. *Fluid Mechanics*. 5th edition. Elsevier, 2011.
2. [**2**] F. Mellibovsky, A. Meseguer. *A mechanism for streamwise localization of nonlinear waves in shear flows*. JFM RAPIDS, vol 779, R1. 2015.
3. [**3**] Y.A. Kuznetsov. *Elements of Applied Bifurcation theory*. 2nd edition. Springer, 2000.