MEIO/UPC-UB : NETWORK FLOWS

Network Flows

UPCOPENCOURSEWARE number 34414

Topic 1: Introduction

F.-Javier Heredia

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH

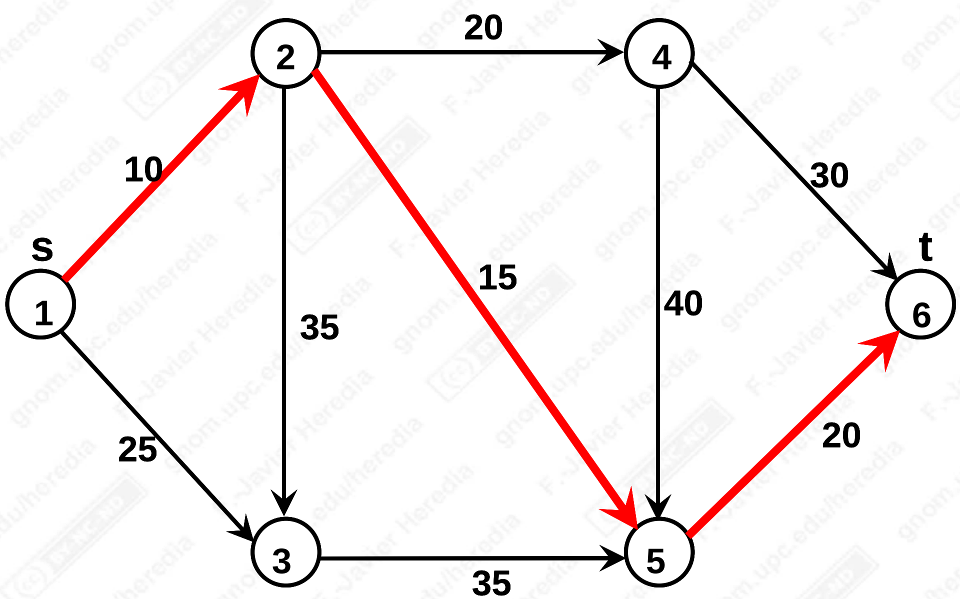
Departament d'Estadística
i Investigació Operativa

Introduction

- Definitions:
  - Minimum shortest path problems.
  - Maximum flow problems.
  - Minimum cost flow problems.
- Network flow problems (NF) as a class of linear programming problems (LP):
  - Standard form of the minimum flow cost problem.
  - Nodes-arcs incidence matrix.
  - Formulation of the minimum shortest path problem.
  - Formulation of the maximum flow problem.
  - Formulation of the transportation and assignment problem.
  - Transformation to the standard form.
- Applications.

Shortest Path Problems

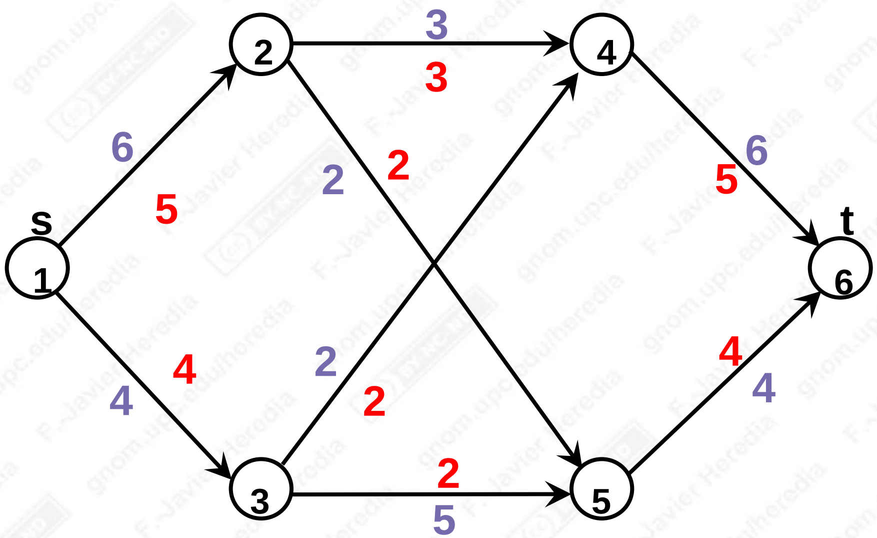
- To Identify the shortest path between a source node ("s") and the sink node ("t").



- Applications:
  - Minimum cost flow problem.
  - Minimum time flow problem.
  - Equipment replacement problems.

Maximum Flow Problems

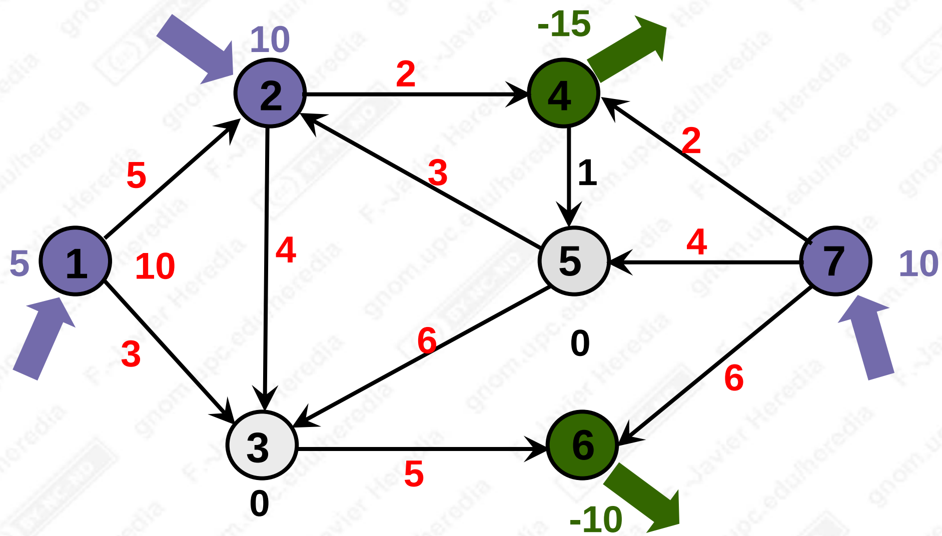
- To seek how to send the maximum possible amount of flow from a source node ("s") to sink node ("t") in a **capacitated network**.



- Applications: determining the maximum steady –state flow of:
  – Petroleum products in a pipeline network.
  – Cars in a road network.
  – Messages in a telecommunication network.
  – Electricity in an electrical network.

Minimum Flow Cost Problems

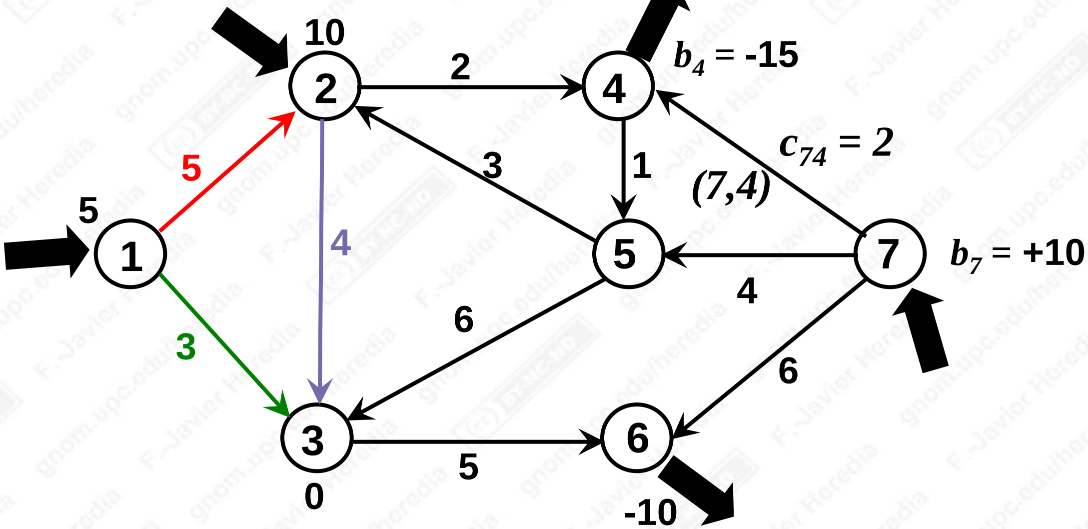
- We wish to determie a least shipment cost of a single commodity through a network in order to satisfy comsuption at **demand nodes** with the production of the **supply nodes**.



- Applications:
  - Logistics(warehouses to retailers).
  - Automobile routing in an urban traffic network.
  - Routing of calls through a telephone system.

Standard form of the minimum cost flow problem (MCNFP) (I)

- Let $G=(N,A)$ be the directed graph defined by the set N of ***n* nodes** ($|N| = n$) and the set A of ***m* directed arcs** ($|A| = m$) .



$n=7$; $m=11$; $N=\{1,2,3,4,5,6,7\}$; $A=\{$ (1,2), (1,3), (2,3),...}

- **Demand/supply vector**:
  b_j , $j=1,2,\ldots,n$: $b=[5,10,0,-15,0,5,10]'$
- **Cost vector:** c_{ij} , $(i,j)\in A$: $c=[5,3,4,\ldots]'$
- **Flow:** x_{ij} , $(i,j)\in A$: amout of commodity to be sent between node i and node j through arc (i,j).

Standard form of the minimum cost flow problem (II)

- Mathematic formulation:

$$
\begin{cases}
\min \quad z = \sum_{(i,j)\in A} c_{ij} x_{ij} & (1.1a) \\
\text{s.a.:} \\
\sum_{\{j:(i,j)\in A\}} x_{ij} - \sum_{\{j:(j,i)\in A\}} x_{ji} = b_i \quad \forall i \in N & (1.1b) \\
x_{ij} \ge 0 \quad \forall (i,j) \in A & (1.1c)
\end{cases}
$$

Balance equations

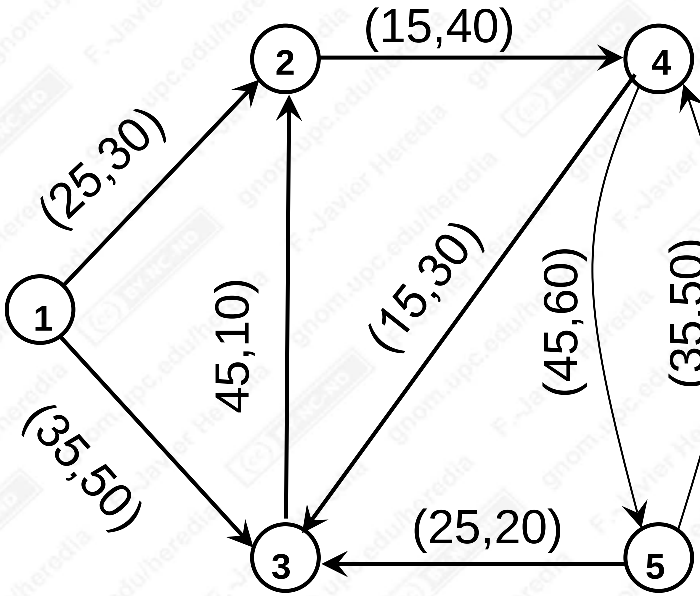
  - We assume the network is **balanced**: $\sum_{i=1}^{n} b_i = 0$

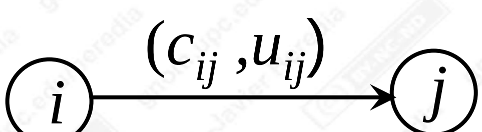
- Matrix notation:

$$
\begin{cases}
\min \quad c'x & (1.2a) \\
\text{s.a.:} \quad T\,x = b & (1.2b) \\
x \ge 0 & (1.2c)
\end{cases}
$$

Node-arc incidence matrix

- Consider the following network:



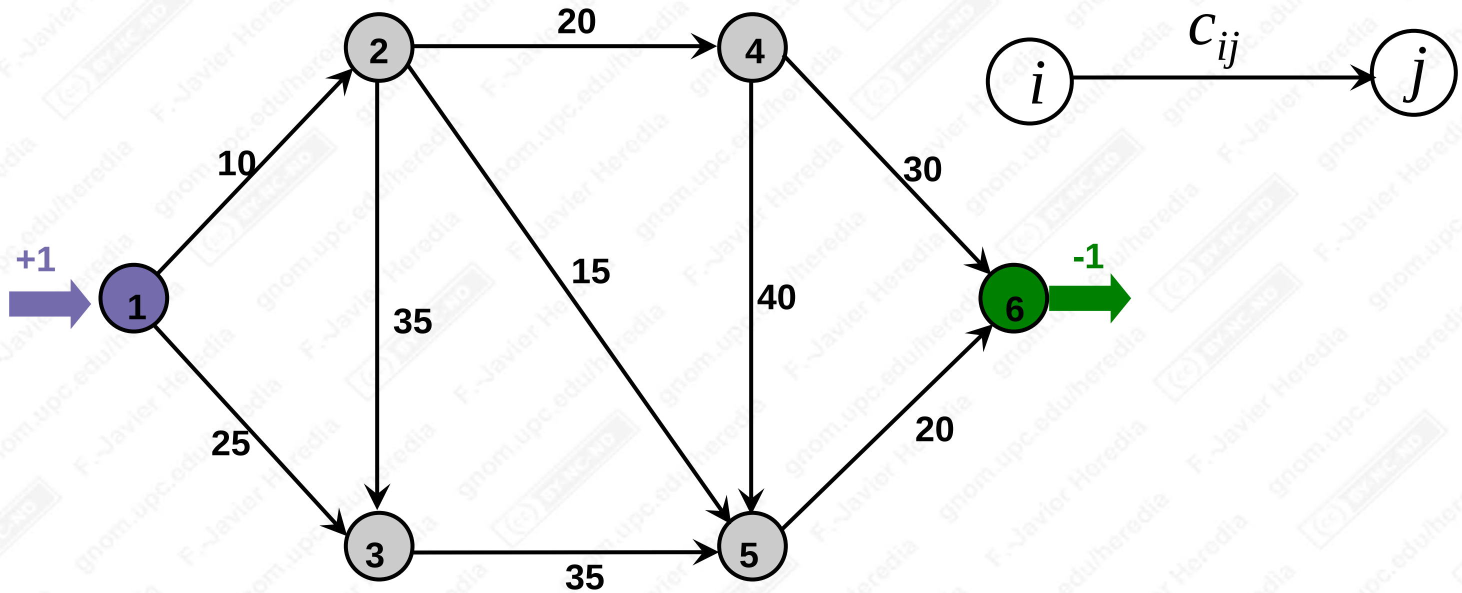


- The associated node-arc incidence matrix T is:

$$
T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{cccccccc} (1,2) & (1,3) & (2,4) & (3,2) & (4,3) & (4,5) & (5,3) & (5,4) \\ \end{array}
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & -1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 1
\end{bmatrix}
$$

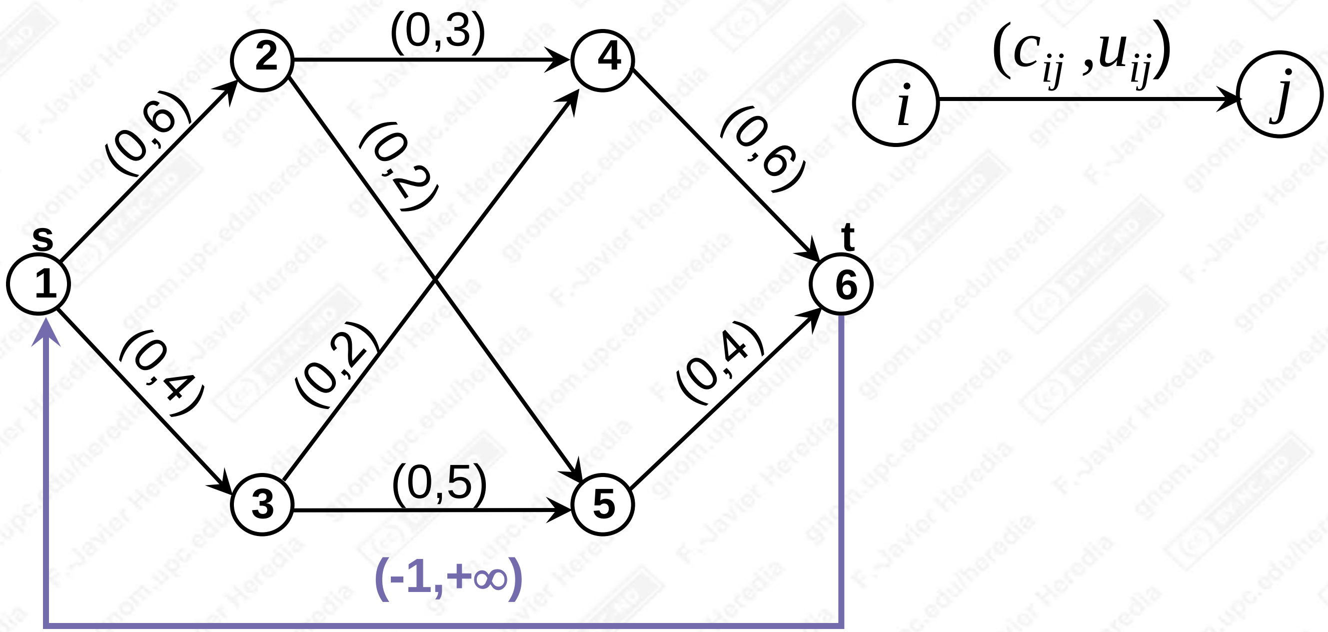
- Properties:
  - $2m$ non-zero elements among nm.
  - Only 2 elements ≠0 per column.
  - Every non-zero elements is +1 or -1.
  - Rank(T)= n-1.

Shortest path problem formulated as a MCNFP



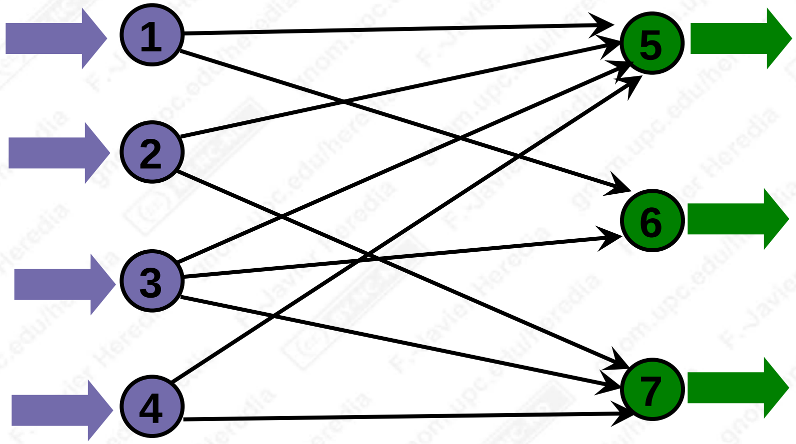
- $b_1 = +1$; $b_6 = -1$; $b_j = 0 \; \forall j \neq 1,6$

Maximum flow problem formulated as a MCNFP



- Artificial arc x_{ts} with c_{ts}=-1 and u_{ts}= +∞.
- $c_{ij}=0\ \forall(i,j)\neq(t,s)$

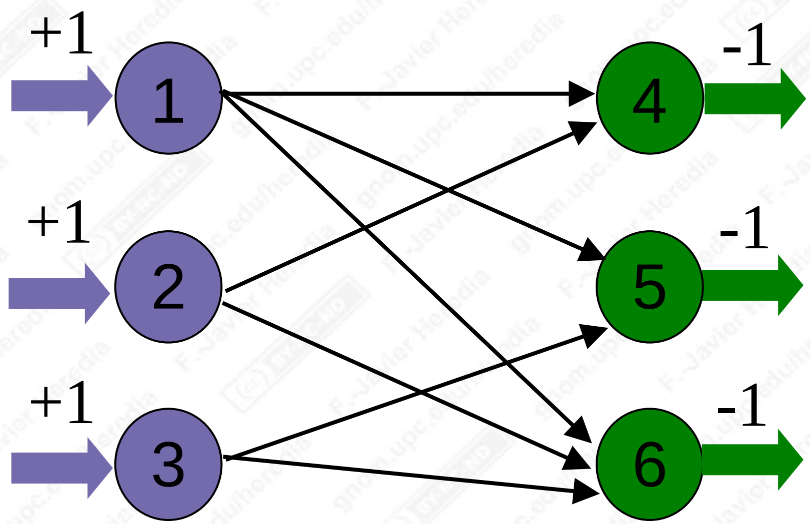
Transportation problem formulated as a MCNFP



- Properties:
  - $N = N_1 \cup N_2$:

    N_1: production nodes; N_2: demand nodes.
  - $\forall (i,j) \in A : i \in N_1 \,;\, j \in N_2$

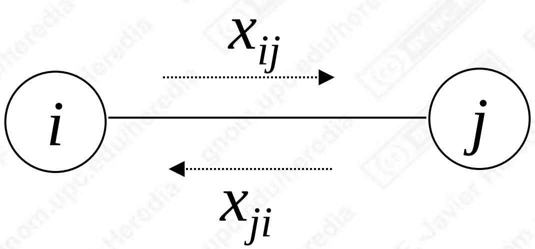
Assignment problem formulation



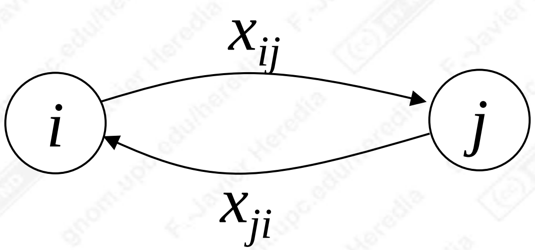
- Properties:
  - $N = N_1 \cup N_2$; $|N_1| = |N_2|$
  - $A \subseteq N_1 \times N_2$.
  - $b_j = +1 \ \forall j \in N_1$; $b_j = -1 \ \forall j \in N_2$

Transformation to the standard form (I)

- **Undirected arcs**:



  - Contribution to the objective function: $c_{ij}x_{ij} + c_{ij}x_{ji}$
  - $c_{ij} \geq 0 \Rightarrow$ at the optimal solution $x_{ij} > 0$ or $x_{ji} > 0$
  - Transformation: each directed arc $\{i,j\}$ is replaced by two directed arcs (i,j) and (j,i) with cost c_{ij} :



Transformation to the standard form (II)

- **Arcs with non-zero lower bounds:**
  - $x_{ij} \geq l_{ij}$
  - x_{ij} is replaced by $x'_{ij}+l_{ij}$ in the problem's formulation:
    - Bounds: $x_{ij} \geq l_{ij} \Rightarrow x'_{ij}+l_{ij} \geq l_{ij}$; $x'_{ij} \geq 0$
    - Balance equations: $b_i \rightarrow b_i - l_{ij}$; $b_j \rightarrow b_j + l_{ij}$
    - Objective function: $c_{ij} x_{ij} \longrightarrow c_{ij} (x'_{ij} + l_{ij}) = c_{ij} x'_{ij} + c_{ij} l_{ij}$

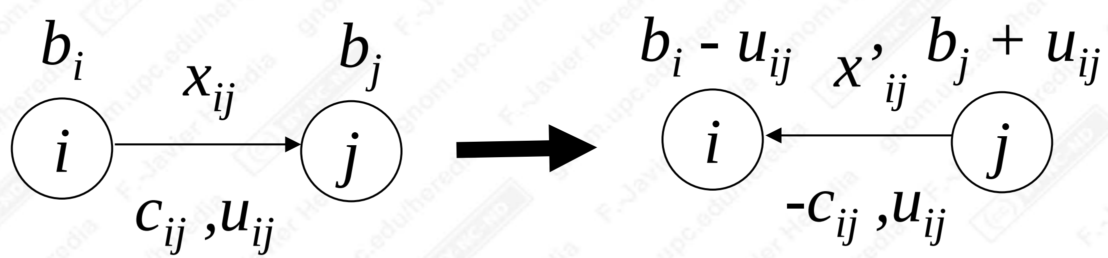
cte.

Transformation to the standard form (III)

- **Arcs with negative costs:**
  - Let be x_{ij} with $c_{ij}<0$.
  - Let u_{ij} a trivial upper bound of the arc (i,j).
  - x_{ij} is replaced by $x'_{ij} = u_{ij} - x_{ij}$.
    - $0 \le x'_{ij} \le u_{ij}$
    - Objective function: (cst) (>0)

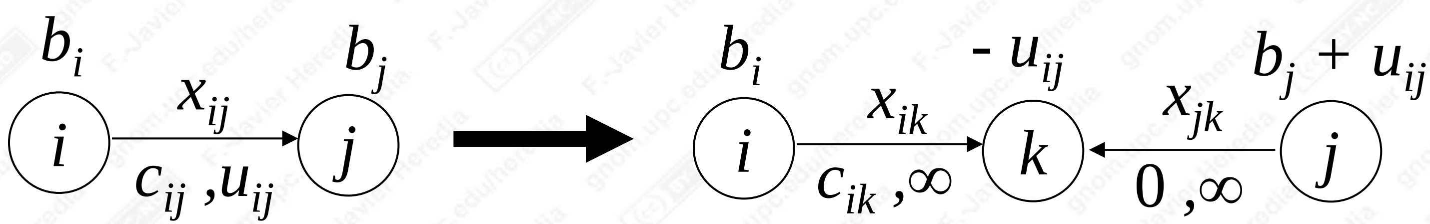
$$z = \ldots + c_{ij}\, x_{ij} + \ldots \longrightarrow z' = \ldots + \underbrace{c_{ij}u_{ij}}_{\text{cst}} \underbrace{- c_{ij}}_{>0} x'_{ij} + \ldots$$

    - Balance equations:



Transformation to the standard form (IV)

- **Arcs with capacity:**
  - Let $0 \le x_{ij} \le u_{ij}$.
  - Network transformation:



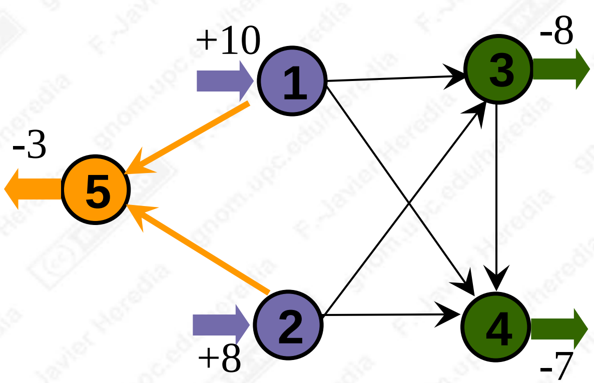
❖ The network is equivalent to the original:

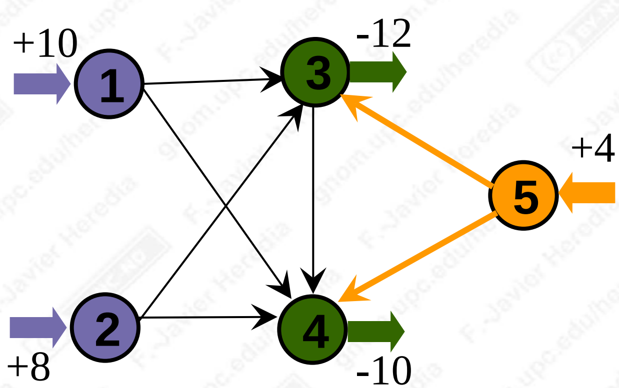
$$x_{ij} \equiv x_{ik} \,;\, x_{ik} + x_{jk} = u_{ij} \Rightarrow x_{ik} = x_{ij} \le u_{ij}$$

❖ The objective function id not modified: $c_{ik} \equiv c_{ij}$

Transformation to the standard form (V)

- **Unbalanced network:**
  - **Excess of production** $\left(\sum_{j=1}^{n} b_j > 0 \right)$:
  - **Excess of demand** $\left(\sum_{j=1}^{n} b_j < 0 \right)$:



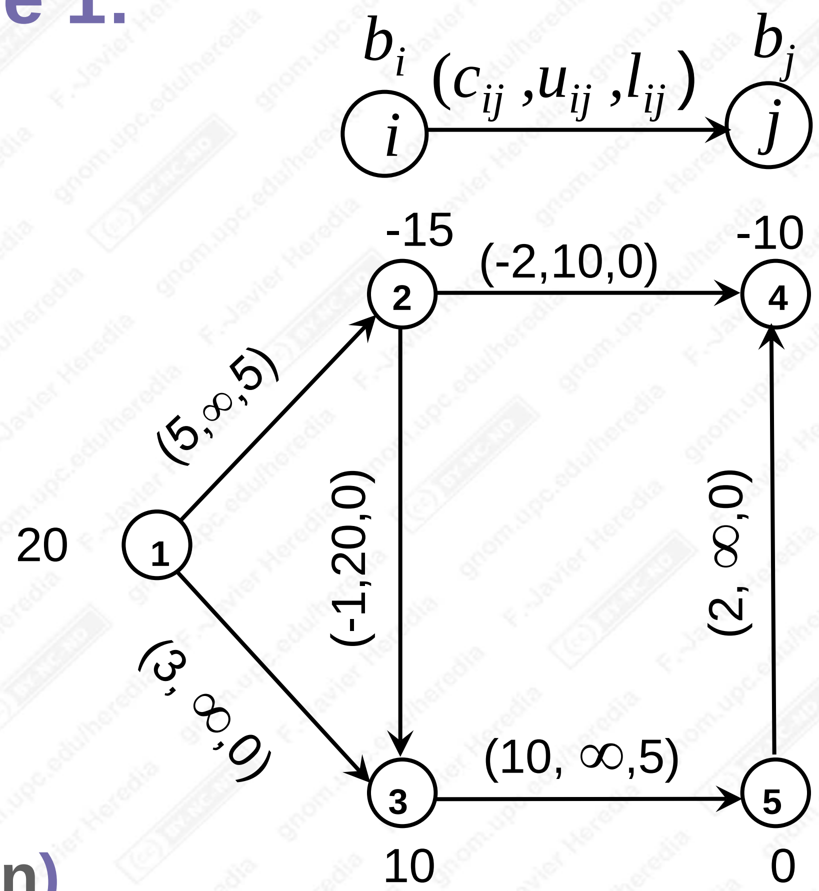


A **dummy demand node** ***n+1*** is added linking all production nodes through **uncapacitated - null cost arcs**.

A **dummy supply node** ***n+1*** is added, linking all demand nodes through **uncapacitated – null cost arcs**.

Transformation to the standard form (VI)

- **Example 1:**



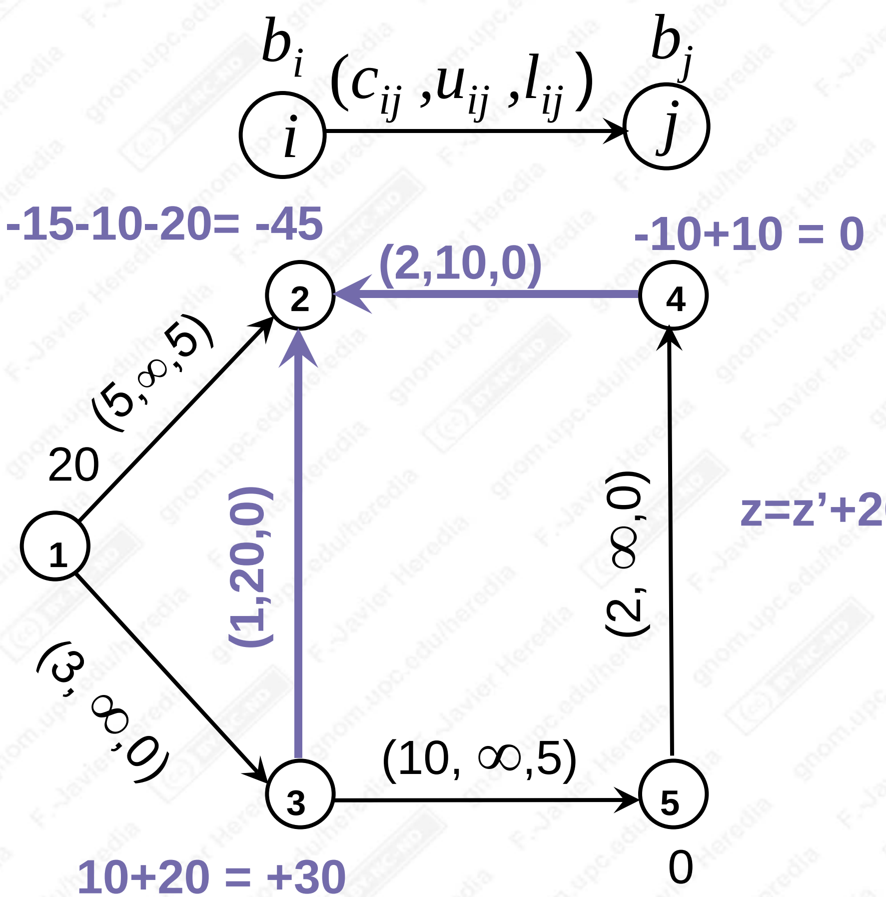
(Solution)

Applications

- Solve the applications examples from book chapter 1.3 of AMO (1 week):
  - Problem description.
  - Network formulation and objective function associated to the problem.
  - Problem classification.
- Problems assigment:

Transformation to the standard form Example (1/4)

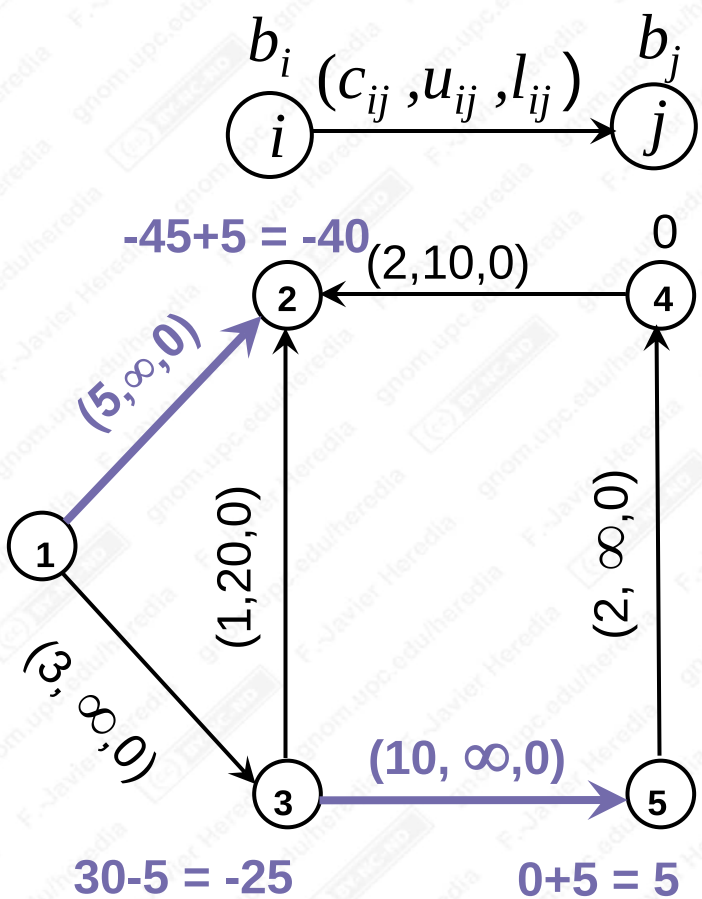
1. Negative cost elimination



z=z'+20+20=z'+40

Transformation to the standard form Example (2/4)

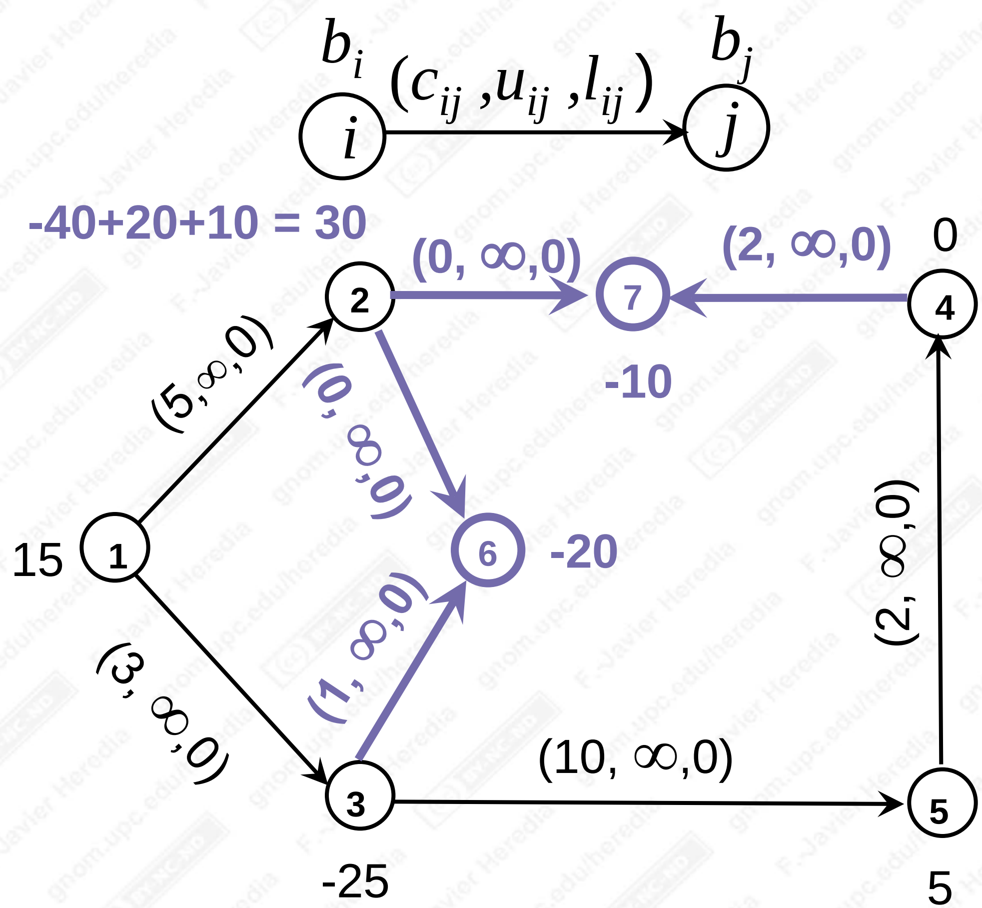
2. Lower bounds elimination ≠0



z=z'+40+25+50=z'+115

Transformation to the standard form Example (3/4)

3. Capacity elimination



z=z'+115

Transformation to the standard form Example (4/4)

4. Unbalance network

z=z'+115