MEIO/UPC-UB : NETWORK FLOWS

Network Flows

UPCOPENCOURSEWARE number 34414

Topic 9: Multicommodity Flows

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9.- Multicommodity flows

- Definitions.
- Applications.
- Properties and optimality conditions.
- Lagrangian relaxation.
- Column generation.
- Source material:
  - R.K. Ahuja, Th.L. Magnanti, J. Orlin "Network Flows", chap. 17.
  - J. Orlin "Network Optimization" http://ocw.mit.edu/courses/sloan-school-of-management/15-082j-network-optimization-fall-2010/

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On the Multicommodity Flow Problem O-D version

K origin-destination pairs of nodes $(s_1, t_1), (s_2, t_2), \ldots, (s_K, t_K)$

Network $G = (N, A)$

d_k = amount of flow that must be sent from s_k to t_k.

u_{ij} = capacity on (i, j) shared by all commodities.

c_{ij}^k = cost of sending 1 unit of commodity k in (i, j).

x_{ij}^k = flow of commodity k in (i, j)

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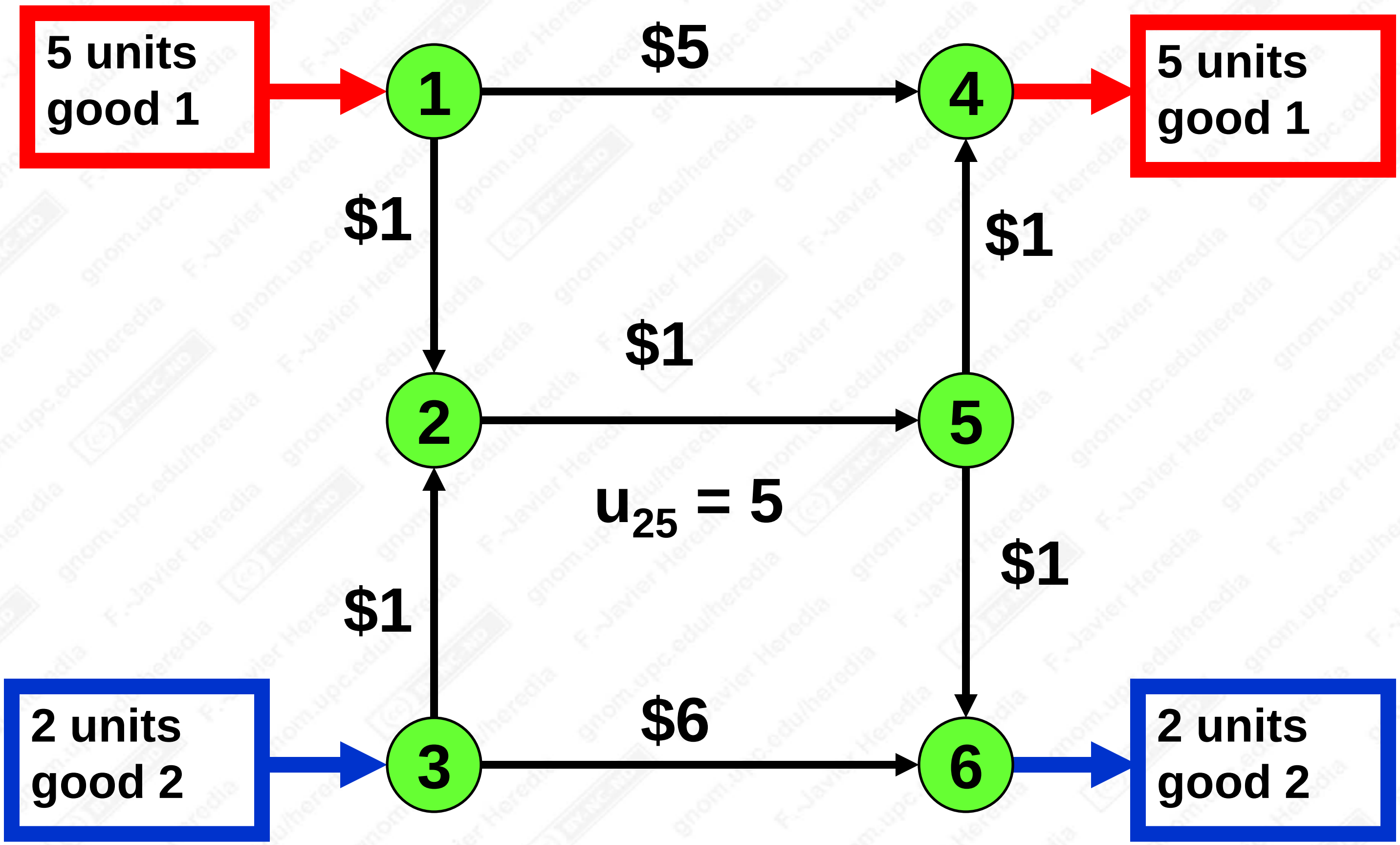
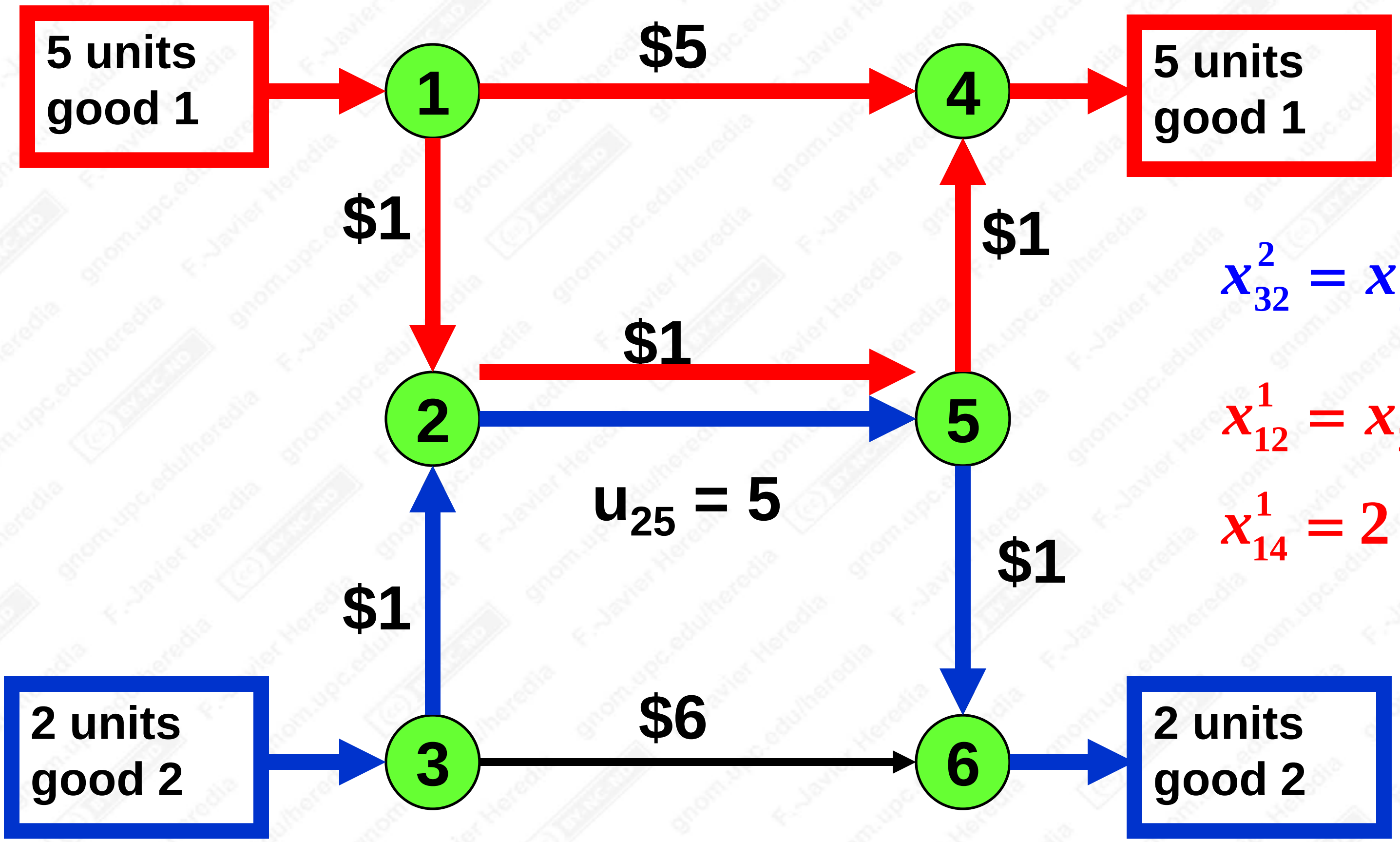
A Linear Multicommodity Flow Problem



Quick exercise: determine the optimal multicommodity flow.

A Linear Multicommodity Flow Problem



$$x_{32}^{2} = x_{25}^{2} = x_{56}^{2} = 2$$

$$x_{12}^{1} = x_{25}^{1} = x_{54}^{1} = 3$$

$$x_{14}^{1} = 2$$

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The Multicommodity Flow LP

$$\text{Min} \sum_{(i,j)\in A} \sum_k c_{ij}^k x_{ij}^k$$

$$\sum_j x_{ij}^k - \sum_j x_{ji}^k = \begin{cases} d_k & \text{if } i = s_k \\ -d_k & \text{if } i \in t_k \\ 0 & \textit{otherwise} \end{cases}$$

Supply/ demand constraints

$$\sum_k x_{ij}^k \le u_{ij} \quad \text{for all } (i,j) \in A$$

Bundle constraints

$$x_{ij}^k \ge 0 \quad \forall (i,j) \in A,\ k \in K$$

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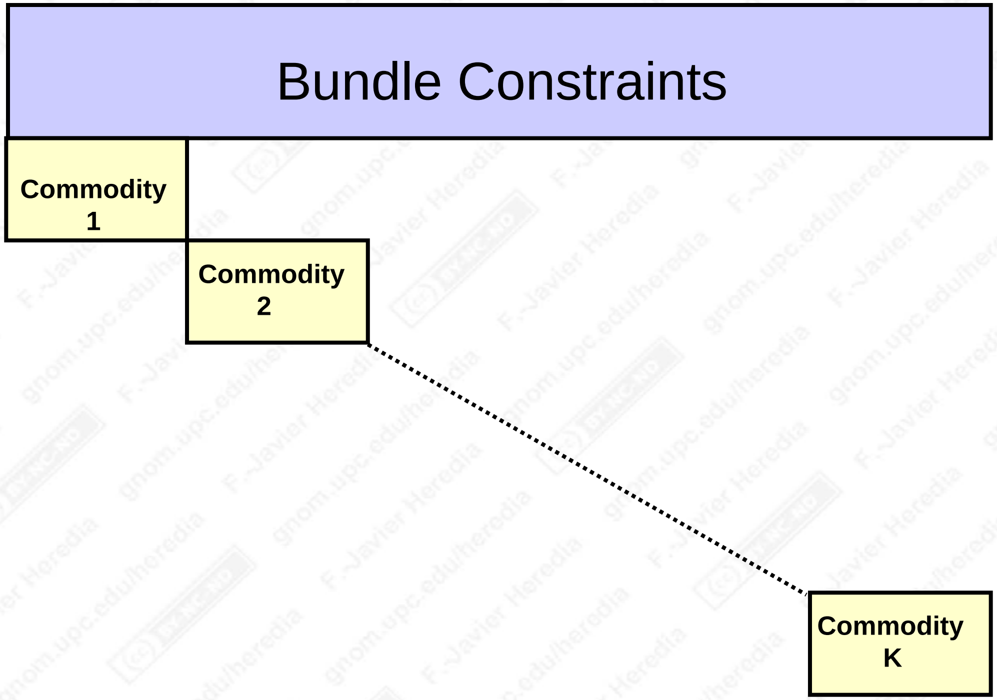
Structure of the Constraint Matrix

- The constraint matrix has the following structure:



- If we can handle the bundle constraints so that they can be ignored, then the multicommodity flow problem decomposes into K independent subproblems.

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Assumptions (for now)

- **Homogeneous goods:** Each unit flow of commodity k on (i,j) uses up one unit of capacity on (i,j).
- **No congestion:** Cost is linear in the flow on (i,j) until capacity is totally used up.
- **Fractional flows:** Flows are permitted to be fractional.
  - In general, multicommodity flow problems have fractional flows, even if all data is integral.
  - The integer multicommodity flow problem is difficult to solve to optimality.
- **OD pairs:** Usually a commodity has a single origin and single destination.

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Application areas

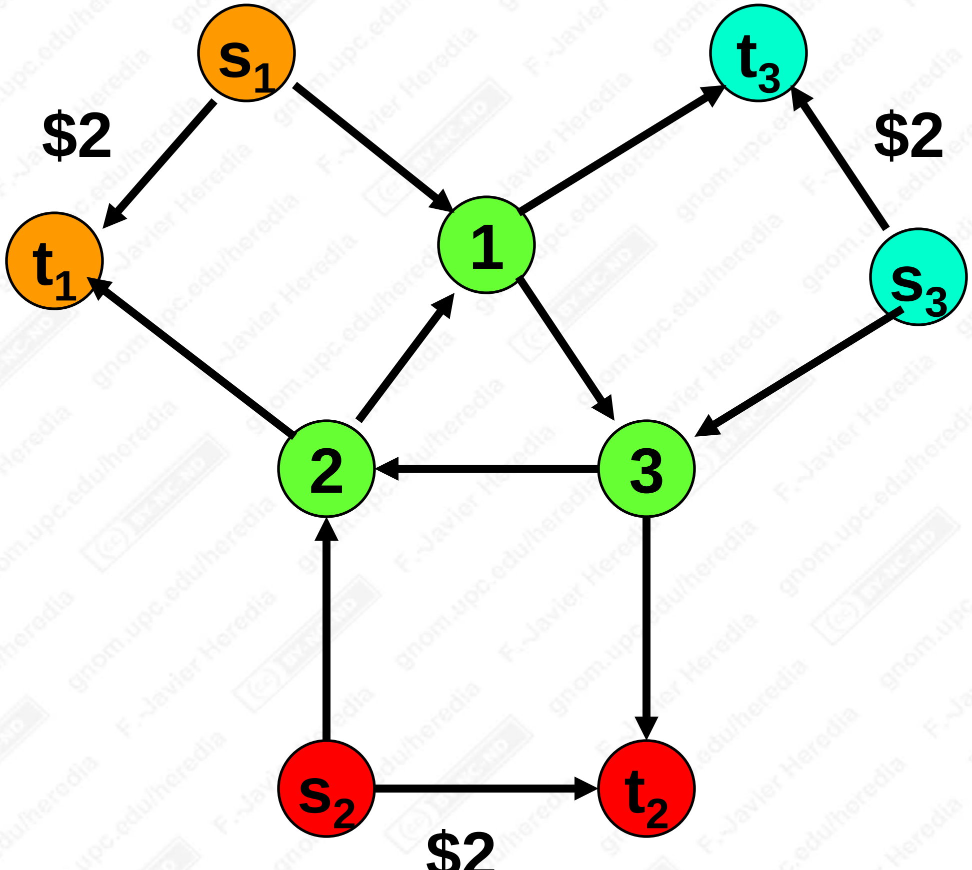
Type of Network	Nodes	Arcs	Flow
Communic. Networks	O-D pairs for messages	Transmission lines	Message Routing
Computer Networks	Storage dev. or Computers	Transmission Lines	Data, Messages
Railway Networks	Yard and Junction pts.	Tracks	Trains
Distribution Networks	Plants, Warehouses,...	Highways Railway Tracks etc.	Trucks, Trains, etc

A fractional multicommodity flow

u_{ij} = 1 for all arcs

c_{ij} = 0 except as listed.

1 unit of flow must be sent from s_i to t_i for i = 1, 2, 3.



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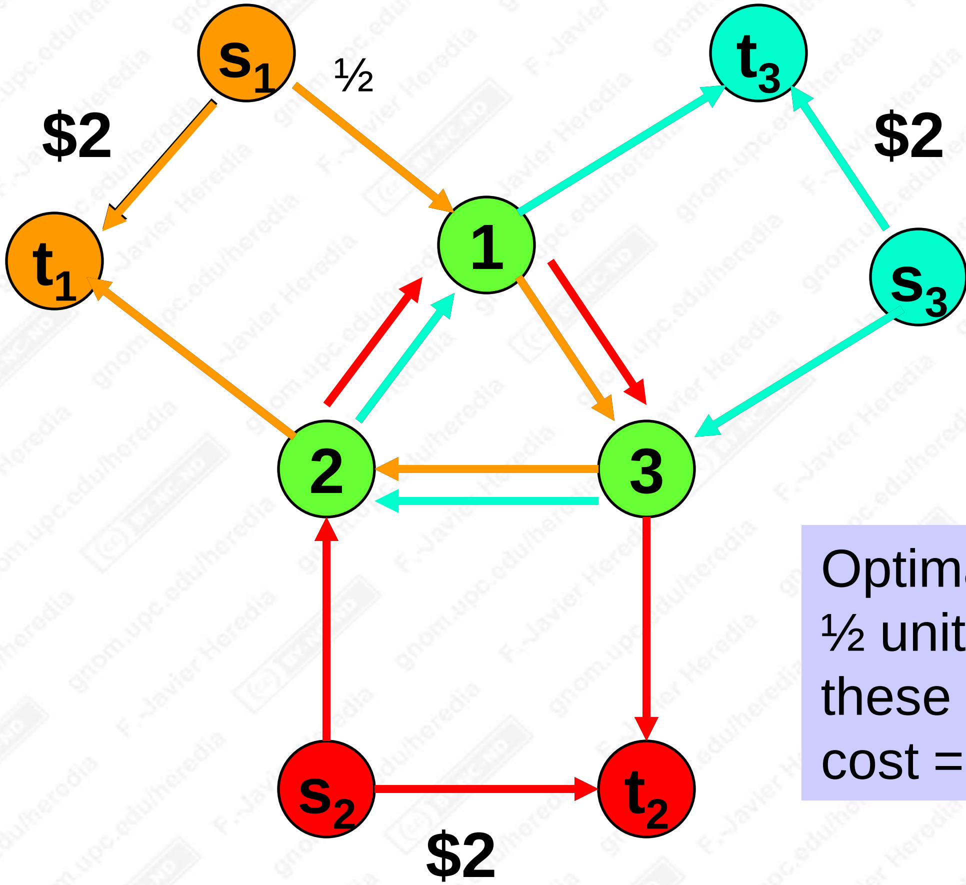


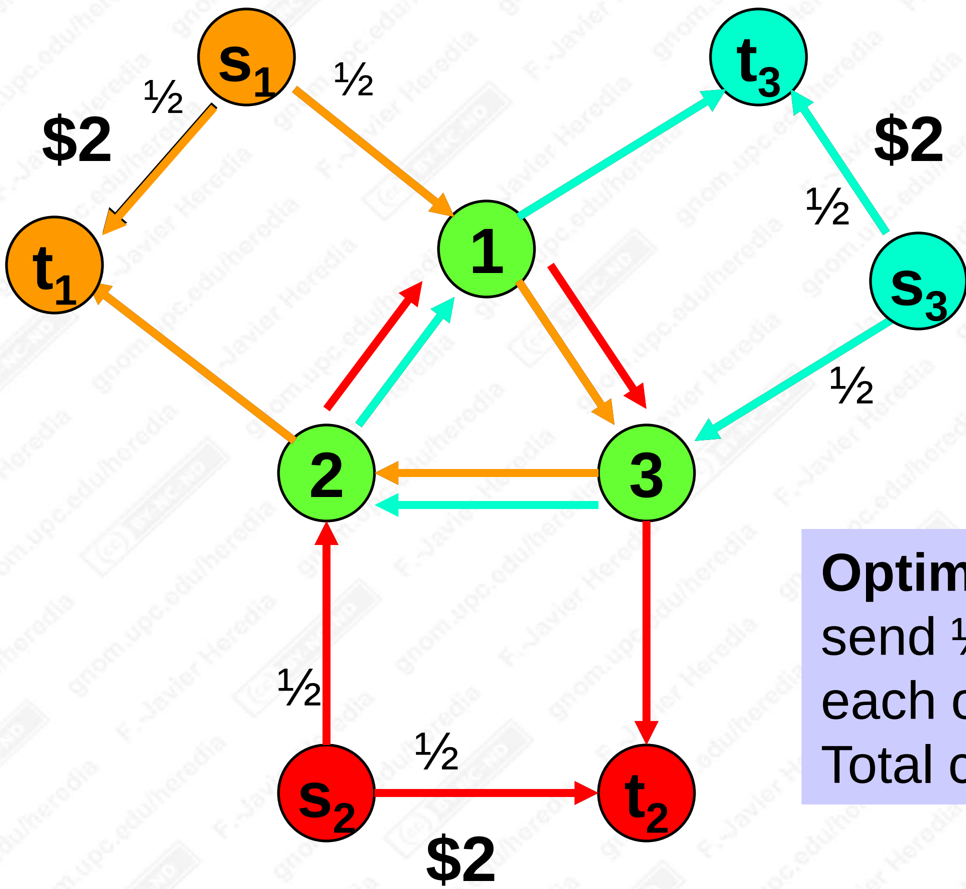
Optimal solution: send ½ unit of flow in each of these 15 arcs. Total cost = $3.

A fractional multicommodity flow

u_{ij} = 1 for all arcs

c_{ij} = 0 except as listed.

1 unit of flow must be sent from s_i to t_i for i = 1, 2, 3.



½

Optimal solution: send ½ unit of flow in each of these 15 arcs. Total cost = $3.

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Decomposition based approaches

Price directed decomposition.

Focus on prices or tolls on the arcs. Then solve the problem while ignoring the capacities on arcs.

Resource directive decomposition.

Allocate flow capacity among commodities and solve

Simplex based approaches

Try to speed up the simplex method by exploiting the structure of the MCF problem.

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A formulation without OD pairs

Minimize $$\sum_{1\le k\le K} c^k x^k \qquad \text{(17.1a)}$$

subject to $$\sum_{1\le k\le K} x_{ij}^k \le u_{ij} \quad \text{for all } (i,j)\in A \qquad \text{(17.1b)}$$

$$Nx^k = b^k \quad \text{for } k = 1, 2, \dots, k \qquad \text{(17.1c)}$$

$$0 \le x_{ij}^k \le u_{ij}^k \quad \begin{array}{l}\text{for all } (i,j)\in A\\ \text{for } k = 1, 2, \dots, k\end{array} \qquad \text{(17.1d)}$$

- The complementary slackness conditions of problem (17.1) can be re-expressed in a set of optimal conditions that avoid the use of the node potentials $\pi^k(j)$ (**partial dualization**).

Optimality Conditions: Partial Dualization

Theorem: *The multicommodity flow $x = (x^k)$ is an optimal multicommodity flow for (17.1) if there exists non-negative prices $w = (w_{ij})$ on the arcs so that the following is true*

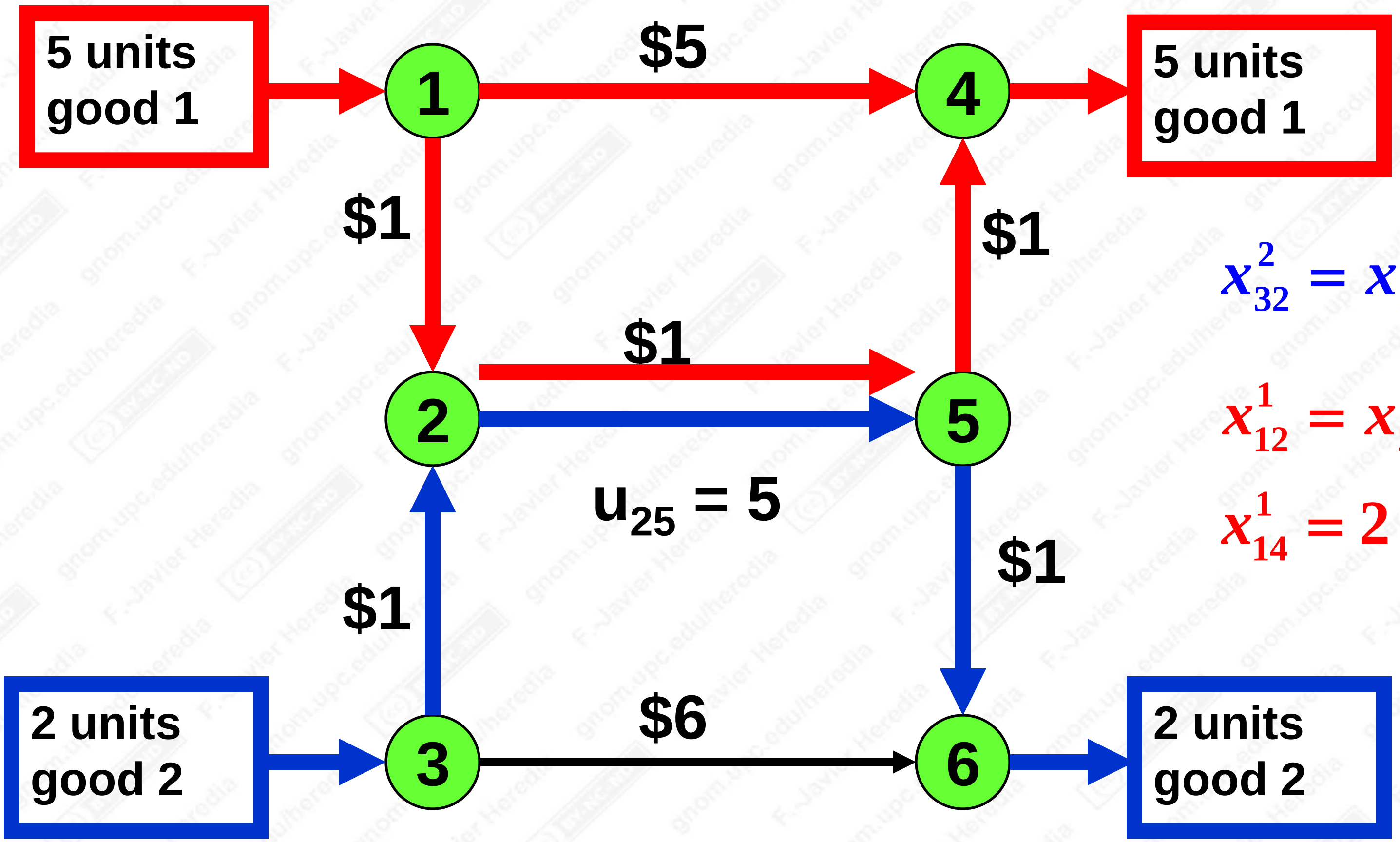
i) $$w_{ij}\left(\sum_k x_{ij}^k - u_{ij}\right) = 0,\ (i,j) \in A\ \left(\equiv\ w_{ij} > 0 \Rightarrow \sum_k x_{ij}^k = u_{ij}\right)$$

ii) *The flow x^k is optimal for the k-th commodity if c^k is replaced by $c^{w,k}$, where*

$$c_{ij}^{w,k} = c_{ij}^k + w_{ij}$$

Recall: x^k is optimal for the k-th commodity if there is no negative cost cycle in the k^{th} residual network.

A Linear Multicommodity Flow Problem



$x^2_{32} = x^2_{25} = x^2_{56} = 2$

$x^1_{12} = x^1_{25} = x^1_{54} = 3$

$x^1_{14} = 2$

Set $w_{2,5}$ = $2

Create the residual networks

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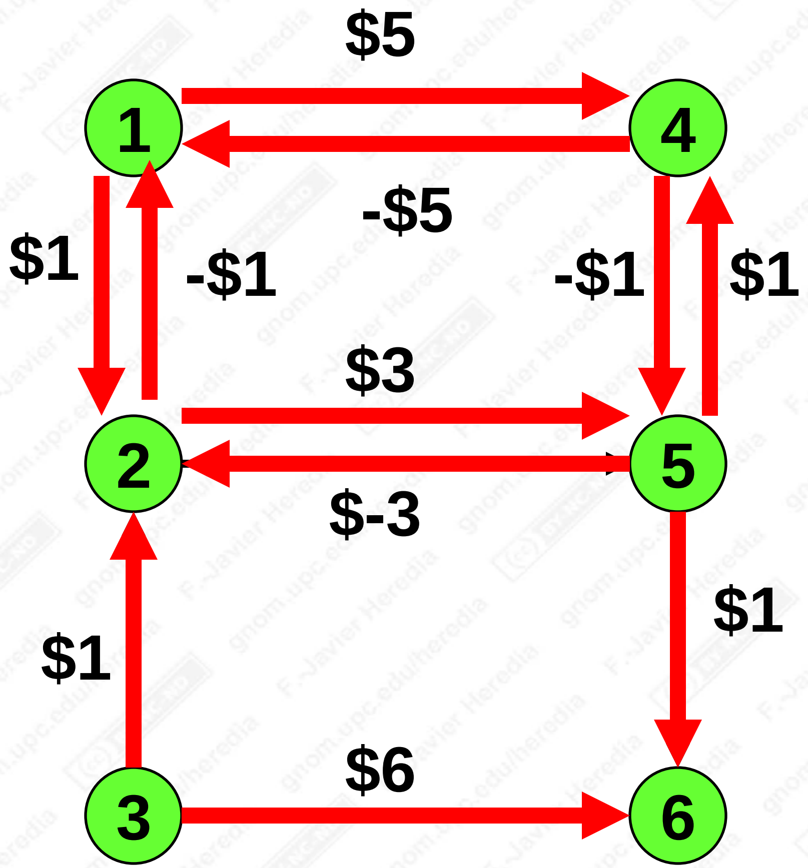
The residual network for commodity 1



Set $w_{2,5}$ = $2

There is no negative cost cycle.

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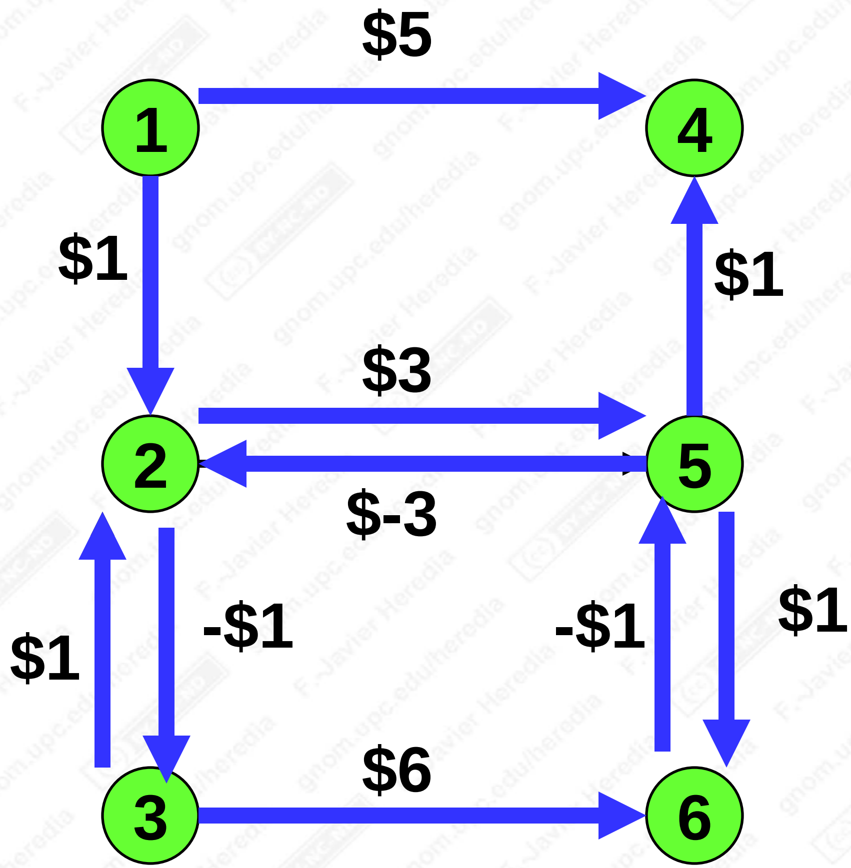
The residual network for commodity 2



Set $w_{2,5}$ = $2

There is no negative cost cycle.

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Optimality Conditions: full dualization

One can also define node potentials π so that the reduced cost

$$c_{ij}^{\pi,k} = c_{ij}^{k} + w_{ij} - \pi_i^k + \pi_j^k \geq 0$$

$$\text{for all } (i,j) \in A \text{ and } k = 1, \ldots, K$$

This combines optimality conditions for min cost flows with the partial dualization optimality conditions for multicommodity flows.

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Lagrangian relaxation for multicommodity flows

Min $$\sum_{(i,j)\in A} \sum_{k} c_{ij}^k x_{ij}^k$$

s.t.:

$$\sum_j x_{ij}^k - \sum_j x_{ji}^k = \begin{cases} d_k & \text{if } i = s_k \\ -d_k & \text{if } i \in t_k \\ 0 & \textit{otherwise} \end{cases}$$

Supply/ demand constraints

$$\sum_k x_{ij}^k \le u_{ij} \quad \text{for all } (i,j) \in A$$

Bundle constraints

$$x_{ij}^k \ge 0 \quad \forall (i,j) \in A,\ k \in K$$

Lagrangian relaxation for multicommodity flows

$$\text{Min} \quad \sum_{(i,j)\in A} \sum_k c_{ij}^k x_{ij}^k + \sum_{(i,j)\in A} w_{ij} \left(\sum_k x_{ij}^k - u_{ij}\right)$$

s.t.:

$$\sum_j x_{ij}^k - \sum_j x_{ji}^k = \begin{cases} d_k & \text{if } i = s_k \\ -d_k & \text{if } i \in t_k \\ 0 & \textit{otherwise} \end{cases}$$

Supply/ demand constraints

$$x_{ij}^k \geq 0 \quad \forall (i,j) \in A, \; k \in K$$

Penalize the bundle constraints.

Relax the bundle constraints.

Lagrangian relaxation for multicommodity flows

Formulate the Lagrangian dual function L(w) (Lagrangian subproblem) and solve the Lagrangian dual

$$(D)\ \max \{ L(w) \mid w \geq 0 \}$$

with:

$$L(w) = \min_{x} \sum_{(i,j)\in A} \sum_{k} (c_{ij}^k + w_{ij}) x_{ij}^k - \sum_{(i,j)\in A} w_{ij} u_{ij}$$

s.t.:

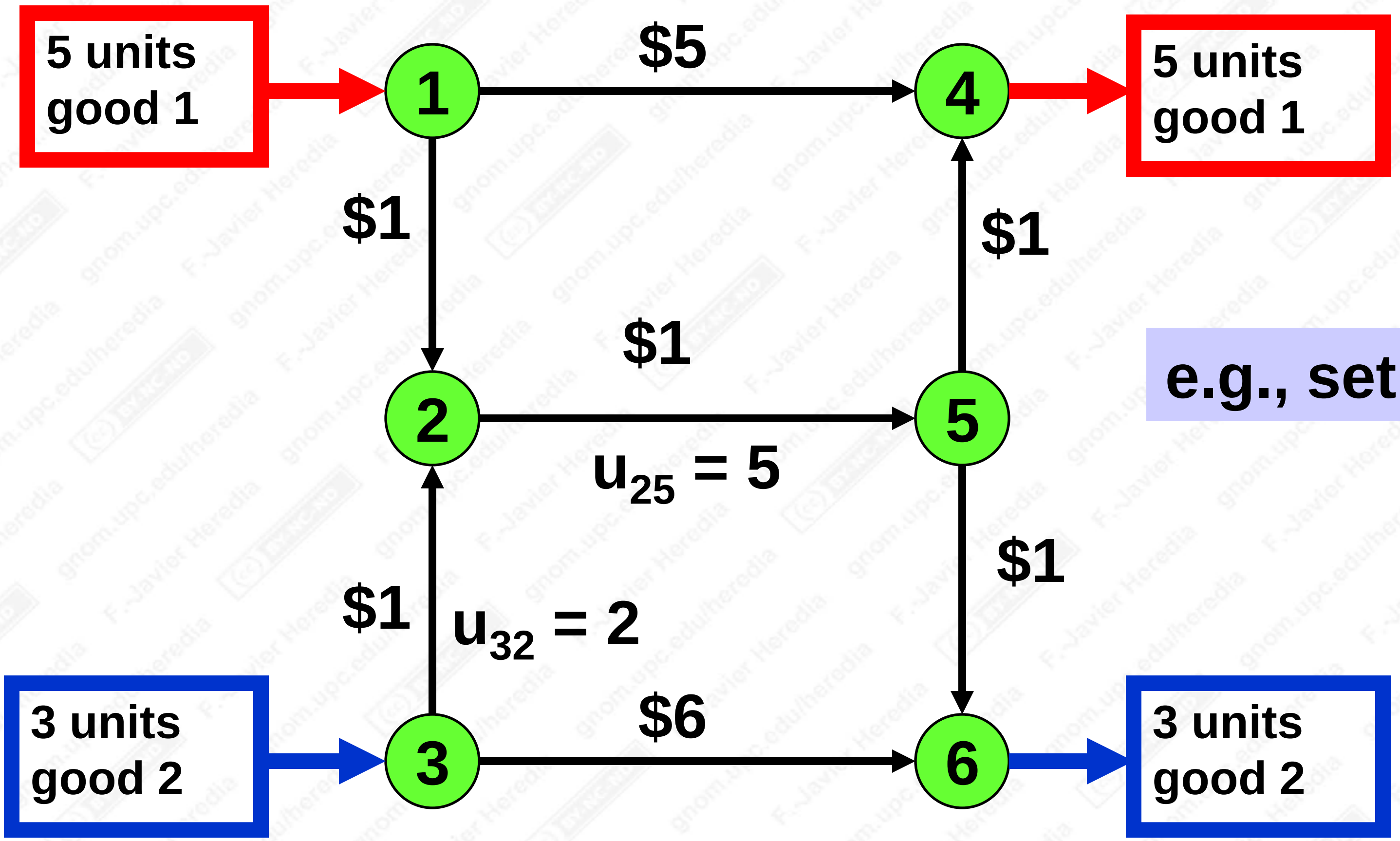
$$\sum_j x_{ij}^k - \sum_j x_{ji}^k = \begin{cases} d_k & \text{if } i = s_k \\ -d_k & \text{if } i \in t_k \\ 0 & \text{otherwise} \end{cases}$$

Supply/ demand constraints

$$x_{ij}^k \geq 0 \quad \forall (i,j) \in A,\ k \in K$$

Remark: observe the relation between the optimality conditions of the Lagrangian dual (D) and the partial dualization optimality conditions.

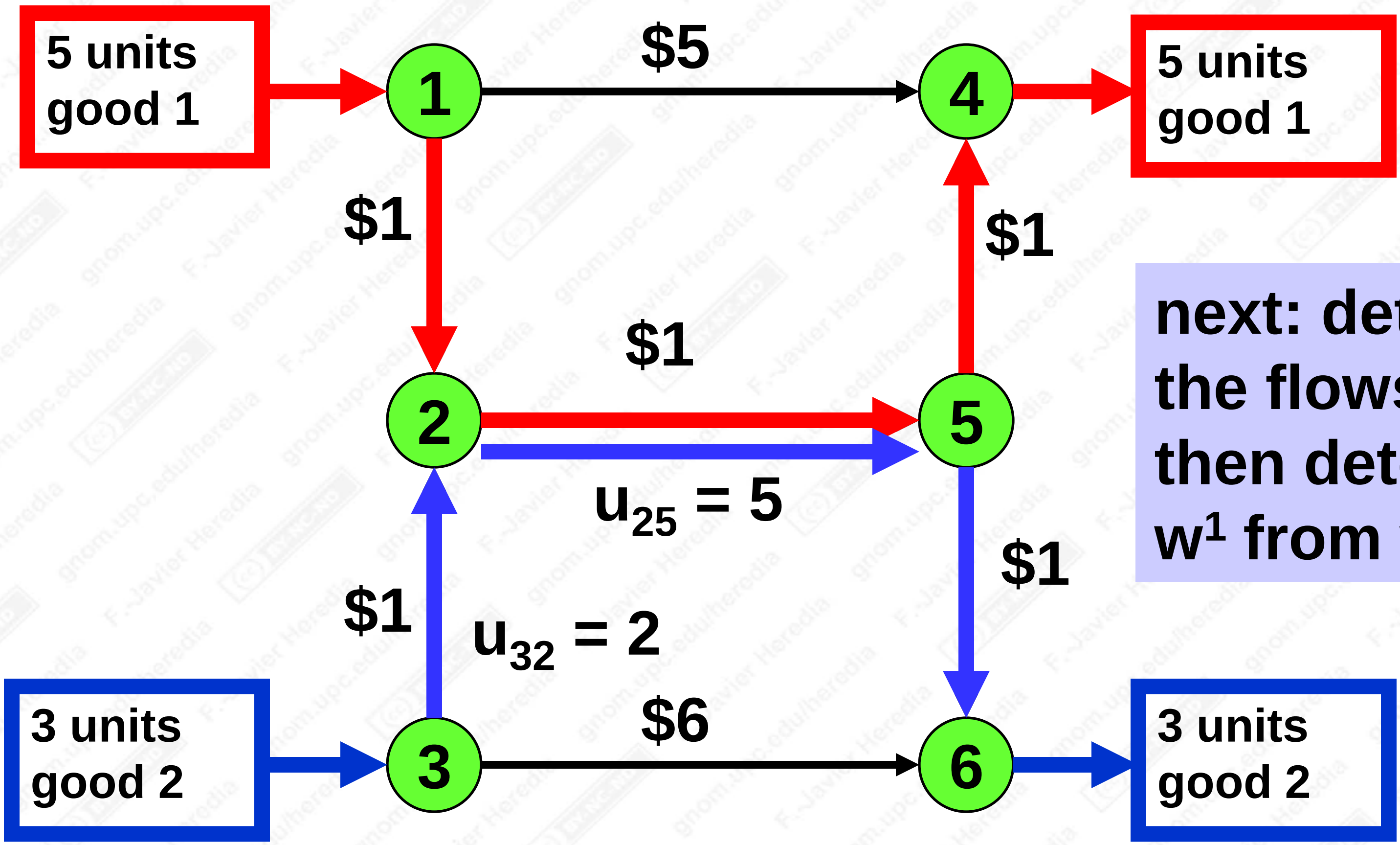
Subgradient Optimization for solving the Lagrangian Multiplier Problem



e.g., set $w^0 = 0$.

Choose an initial value w^0 of the "tolls" w, and find the optimal solution for L(w).

Subgradient Optimization for solving the Lagrangian Multiplier Problem



next: determine the flows, and then determine w^1 from w^0

The flow on (2,5) = 8 > u_{25} = 5.

The flow on (3,2) = 3 > u_{32} = 2.

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Choosing a search direction

$$r^{+} = \max(0, r)$$

$$y_{ij} = \sum_k x_{ij}^k \quad = \text{flow in arc (i,j)}$$

$$w_{ij}^{q+1} = [w_{ij}^{q} + \theta_q (y_{ij} - u_{ij})]^{+}$$

$(y-u)^+$ is called the *search direction*.

$$w_{25}^{1} = [w_{25}^{0} + \theta_0 (8-5)]^{+} = 3\theta_0$$

θ_q is called the *step size*.

$$w_{32}^{1} = [w_{32}^{0} + \theta_0 (3-2)]^{+} = \theta_0$$

So, if we choose $\theta_0 = 1$, then $w_{25}^{1} = 3$ **and** $w_{32}^{1} = 1$

Then solve $L(w^1)$.

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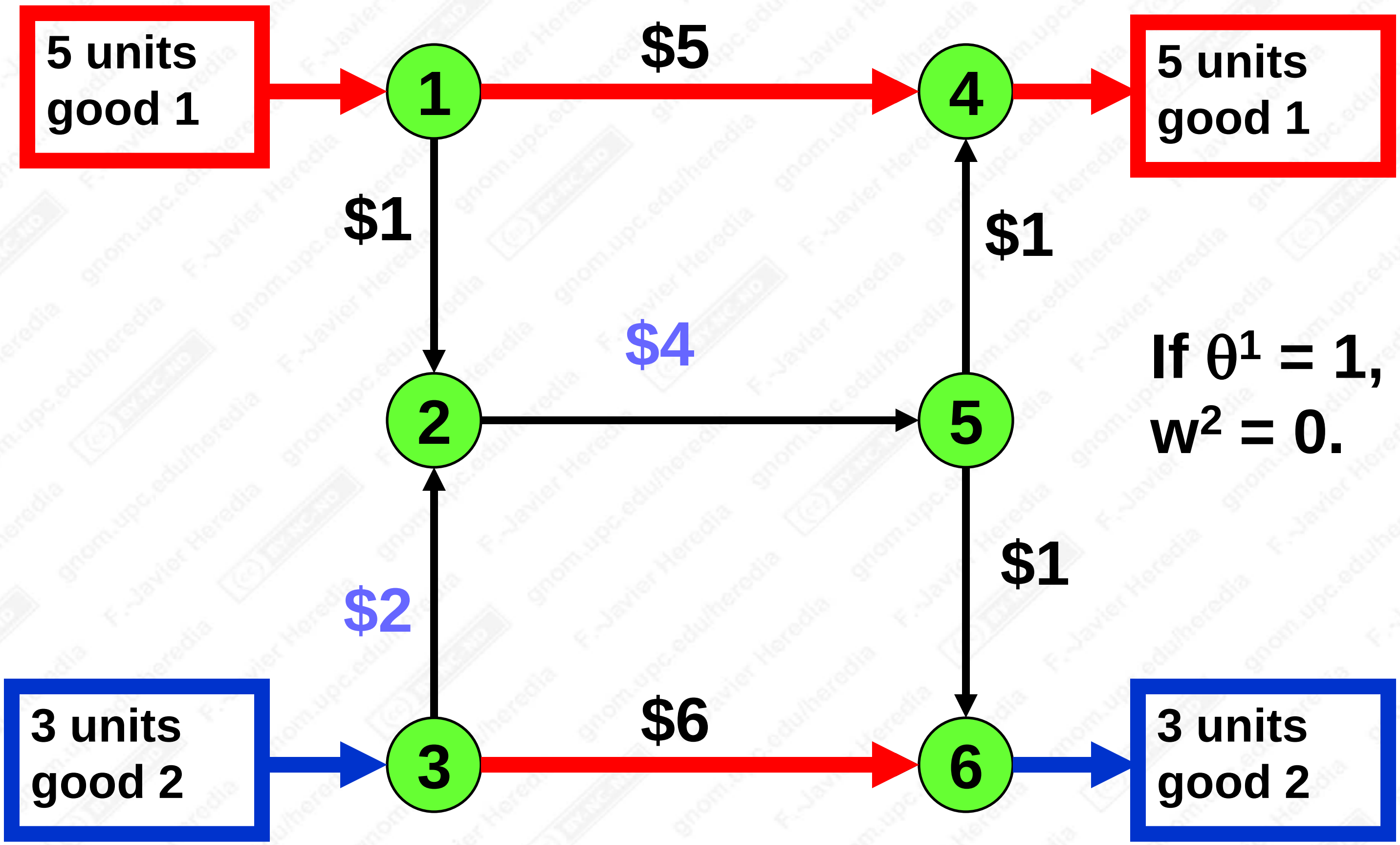
Solving $L(w^1)$



If $\theta^1 = 1$, then $w^2 = 0$.

$$w_{25}^2 = [w_{25}^1 + \theta_1(0-5)]^+ = [3-5\theta_1]^+$$

$$w_{32}^2 = [w_{32}^1 + \theta_1(0-2)]^+ = [1-2\theta_1]^+$$

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Comments on the step size

- The search direction is a good search direction.
- But the step size must be chosen carefully.
- Too large a step size and the solution will oscillate and not converge
- Too small a step size and the solution will not converge to the optimum.

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On choosing the step size

The step size θ_q should be chosen so that

$$\lim_{q \to \infty} \theta_q = 0 \quad \text{and} \quad \sum_{q=1}^{\infty} \theta_q = \infty \qquad (1)$$

e.g., take $\theta_q = 1/q$.

Theorem. *If the search direction is chosen as on the previous slides (**subgradient search**), and if (θ_q) satisfies (1), then the w^q converges to the optimum for the Lagrangian dual problem (D).*

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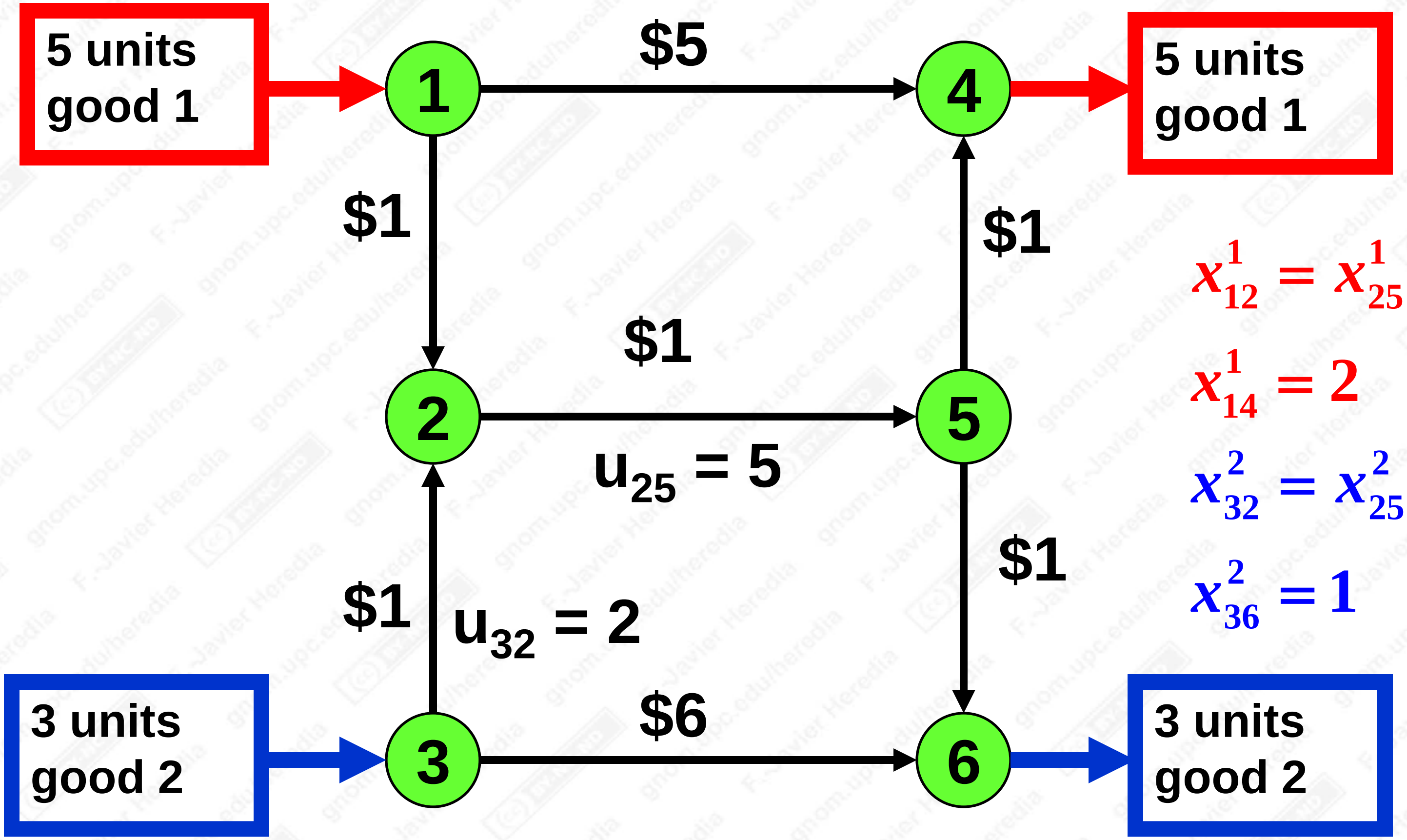
The optimal multipliers and flows.



$$x_{12}^1 = x_{25}^1 = x_{54}^1 = 3$$

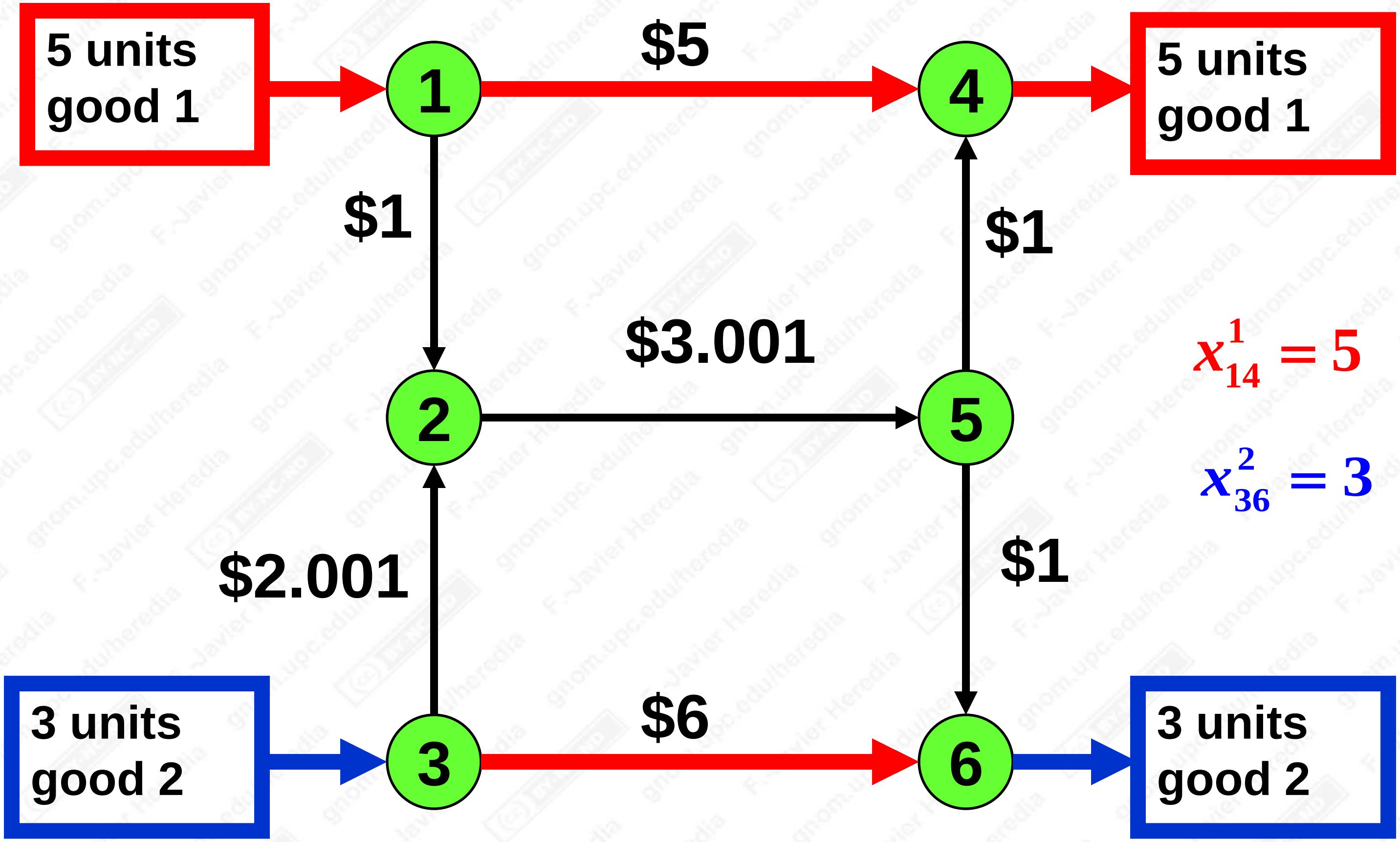
$$x_{14}^1 = 2$$

$$x_{32}^2 = x_{25}^2 = x_{56}^2 = 2$$

$$x_{36}^2 = 1$$

$$\lim_{q\to\infty} w_{32}^q = 1 \qquad \lim_{q\to\infty} w_{25}^q = 2$$

Suppose that w_{32} = 1.001 and w_{25} = 2.001



$x^1_{14} = 5$

$x^2_{36} = 3$

Conclusion: Near Optimal Multipliers do not always lead to near optimal (or even feasible) flows.

Column Generation: Arc-Path Formulation

- The formulation of the multicommodity flow problem we gave earlier is known as the **node-arc formulation**.
- We will now give another formulation known as the **arc-path formulation**. This formulation uses the fact that **any arc flow can be decomposed into flow along paths and cycles** (*Flow decomposition theorem, Th. 3.5 A-M-O*).

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Flow Decomposition Theorem

- **Flow Decomposition Theorem***.*
  Any non-negative feasible flow x can be decomposed into the following:
  i. the sum of flows in paths directed from supply nodes to demand nodes, plus
  ii. the sum of flows around directed cycles.
  It will always have at most n + m paths and cycles.
- **Remark:** The decomposition usually is not unique.

Flow decomposition algorithm

begin
  Initialize
  while y ≠ ∅ **do**
  begin
    Select(s, y)
    Search(s, y)
    if a cycle C is found **then do**
    begin
      let Δ = Capacity(C, y)
      Add Flow(Δ, C) to cycle flows
      Subtract Flow(Δ, C) from y.
    end
    if a path P is found **then do**
    begin
      let Δ = Capacity(P, y)
      Add Flow(Δ, P) to path flows
      Subtract Flow(Δ, P) from y.
    end
end

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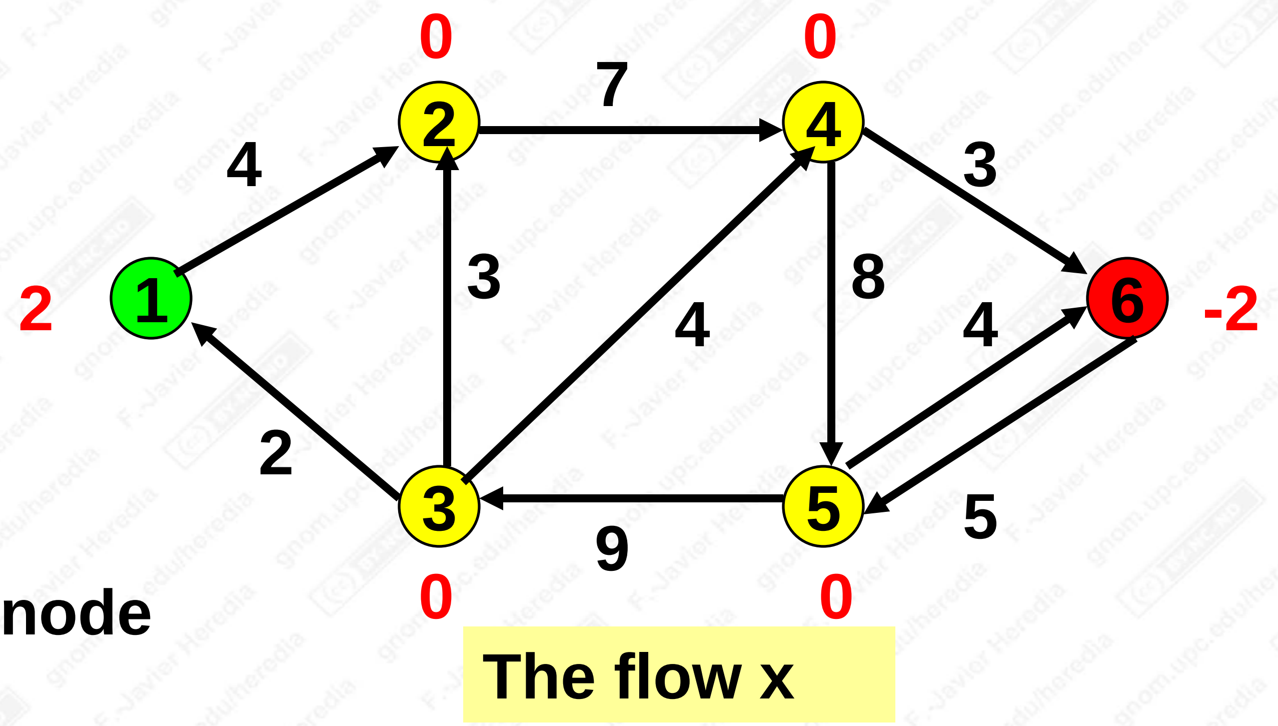
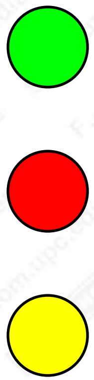
The initial flow





A supply node

A demand node

A balanced node

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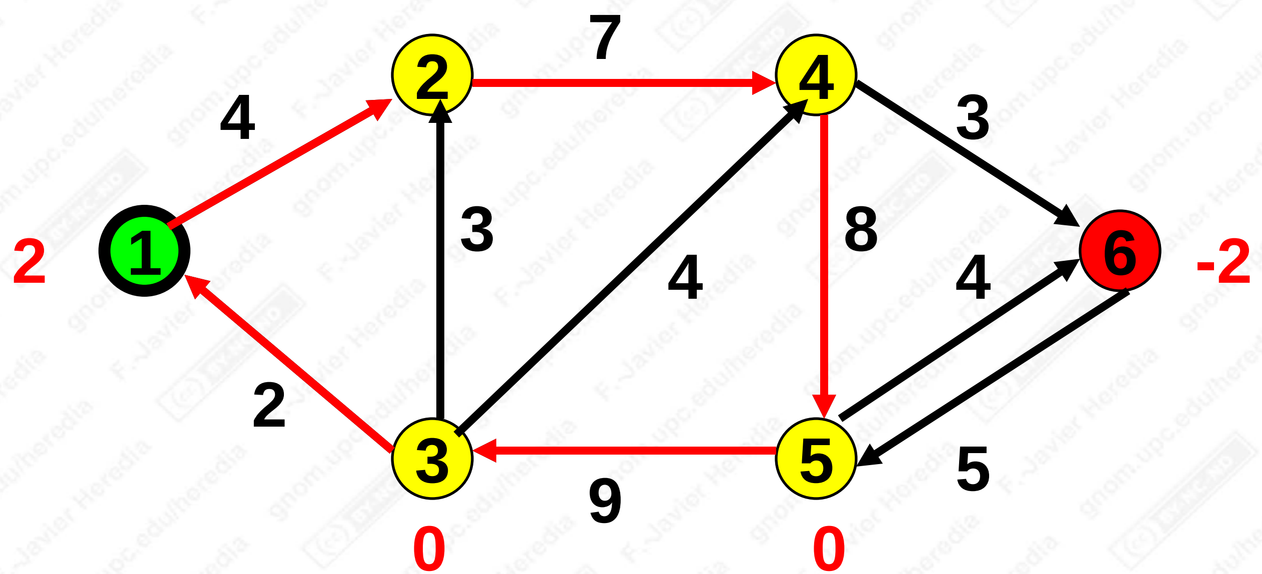
Find a Path or Cycle

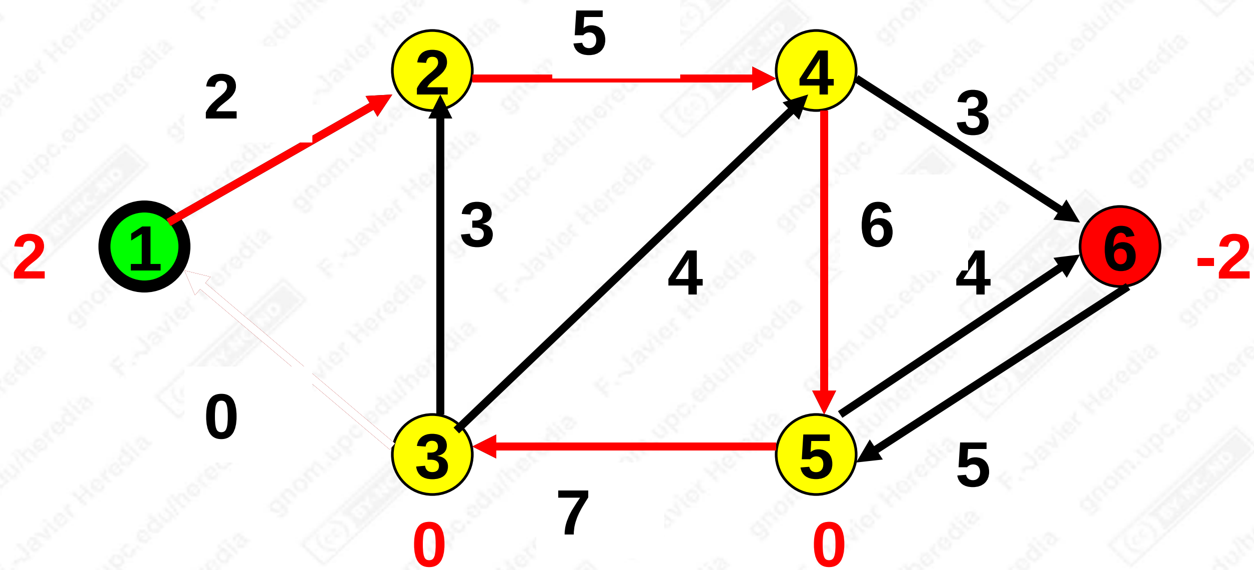


Carry out a depth first search until a cycle C or a path P is found

Determine the capacity

The capacity is 2.

Update the decomposition



Add the cycle flow to the decomposition

update the current flow

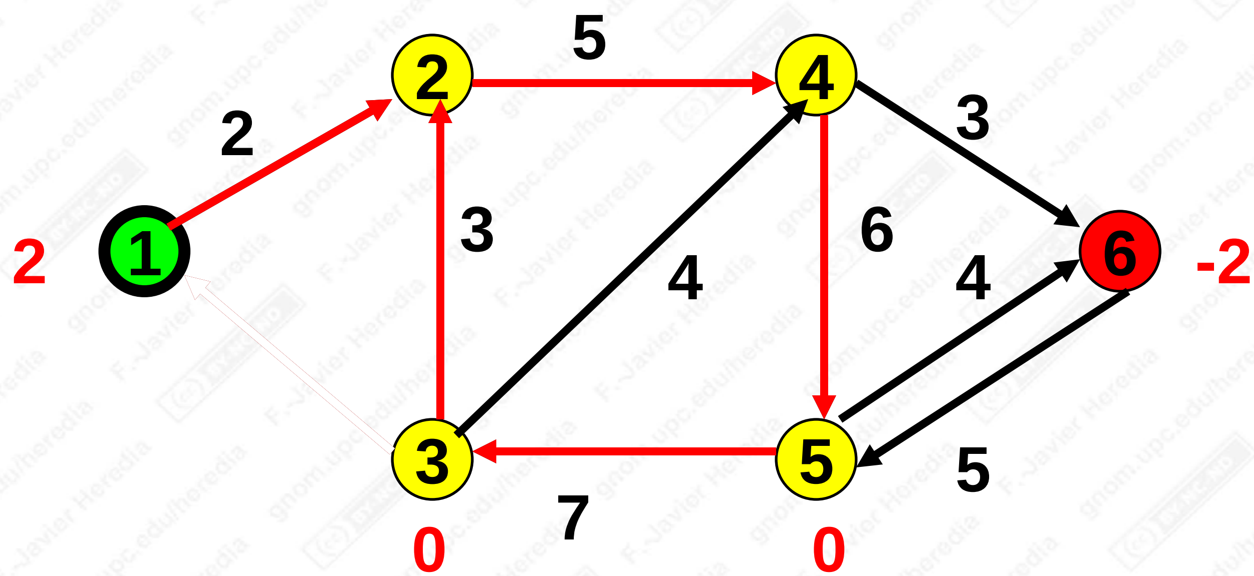
cycle flows

2 units around 1-2-4-5-3-1

path flows

Find the next path or cycle

Start at a supply node and find the next cycle or path



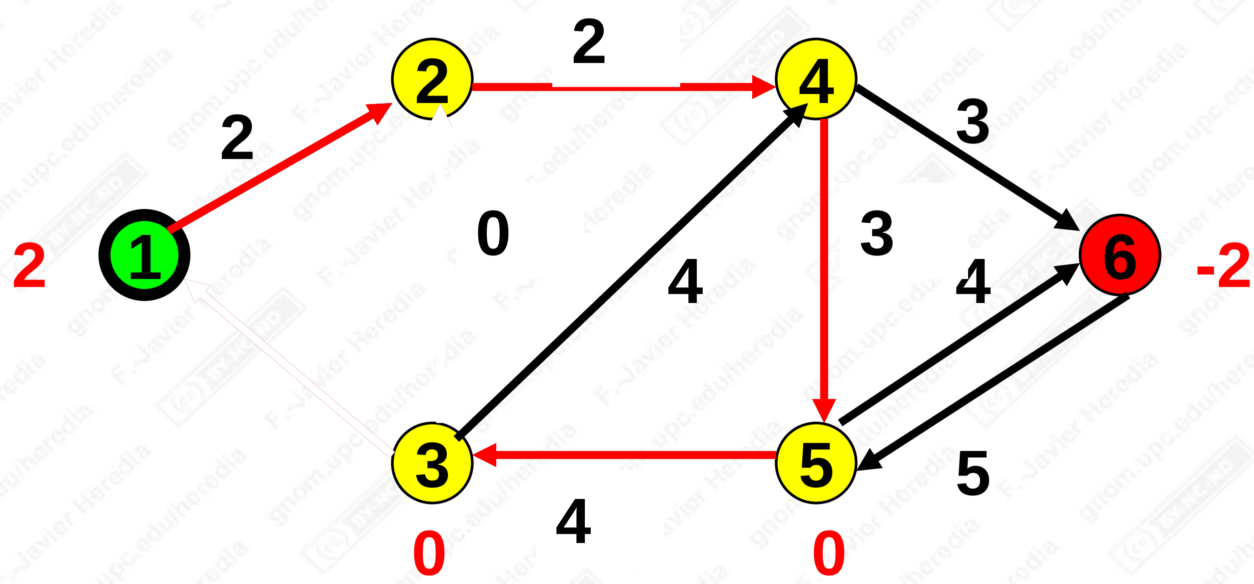
cycle flows

2 units around 1-2-4-5-3-1

path flows

update the decomposition and current flow

update the current flow and decomposition



cycle flows

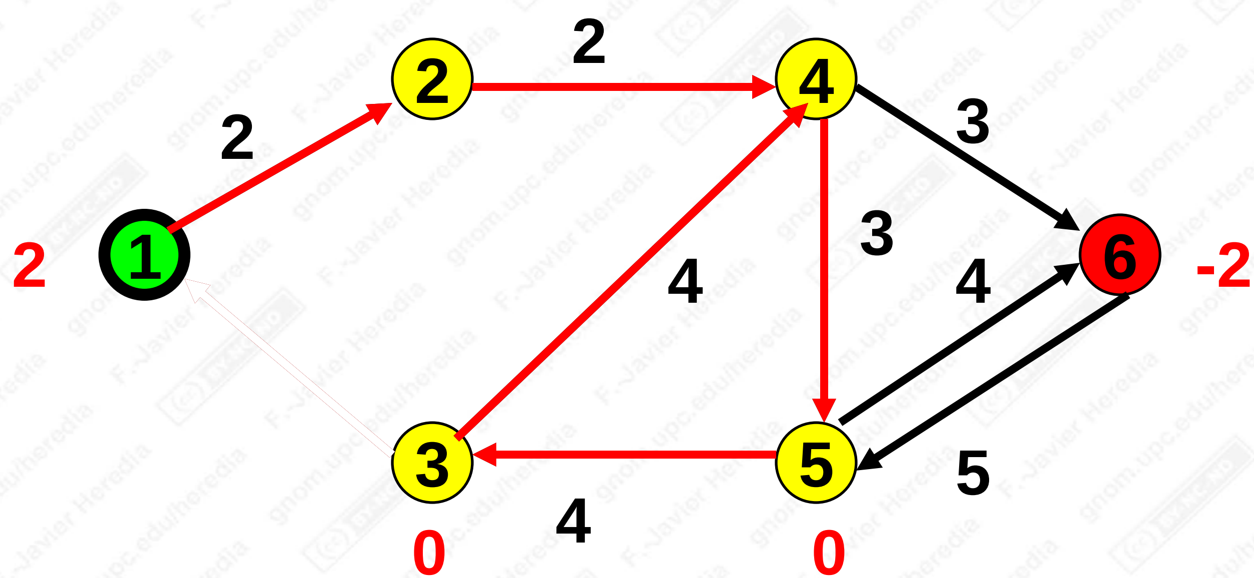
2 units around 1-2-4-5-3-1

3 units around 2-4-5-3-2

path flows

Find the next flow or cycle

start with a supply node and find the next flow or cycle.



The capacity of 4-5-3-4 is 3

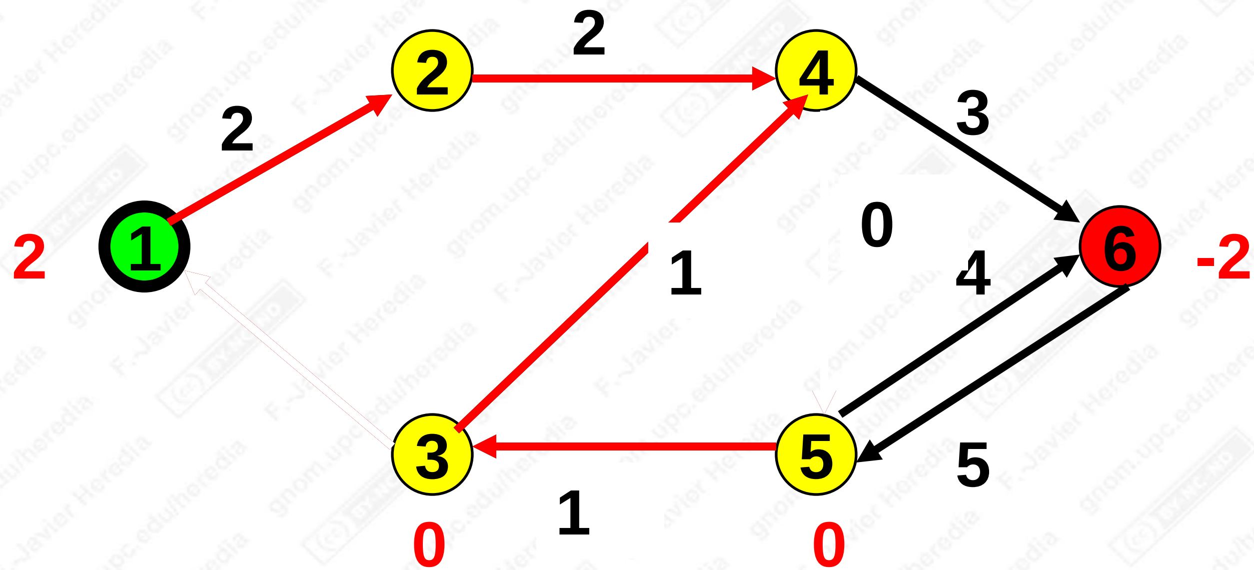
cycle flows

2 units around 1-2-4-5-3-1

3 units around 2-4-5-3-2

path flows

Update



update the current flow and decomposition

cycle flows

2 units around 1-2-4-5-3-1

3 units around 2-4-5-3-2

3 units around 4-5-3-4

path flows

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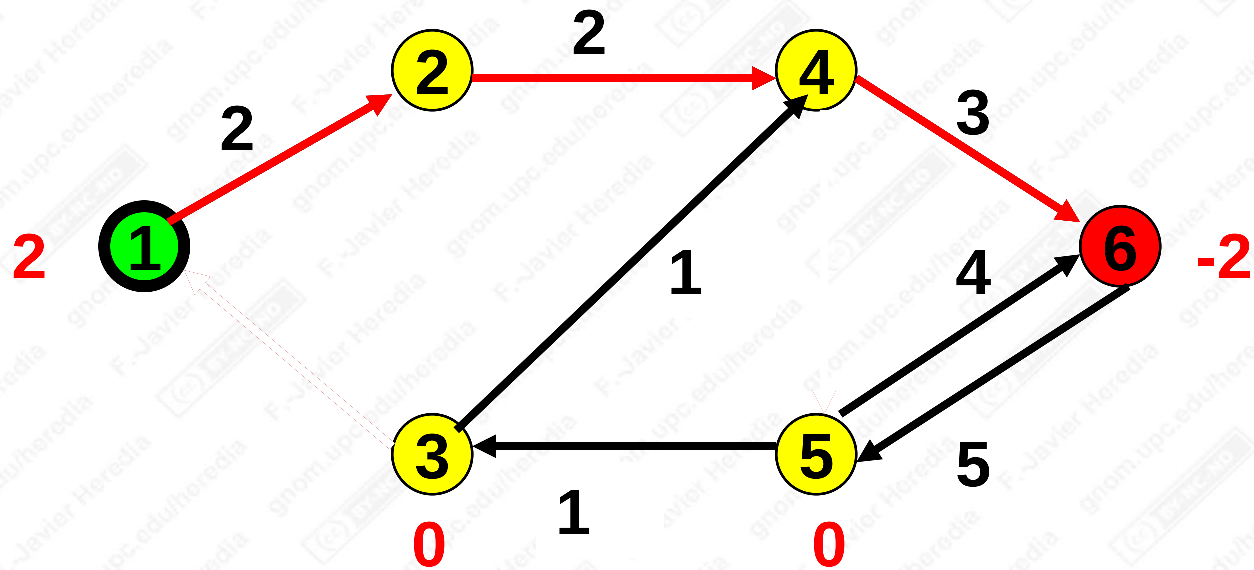
Find the next flow or cycle

start with a supply node and find the next flow or cycle.

The capacity of 1-2-4-6 is 2



cycle flows

2 units around 1-2-4-5-3-1

3 units around 2-4-5-3-2

3 units around 4-5-3-4

path flows

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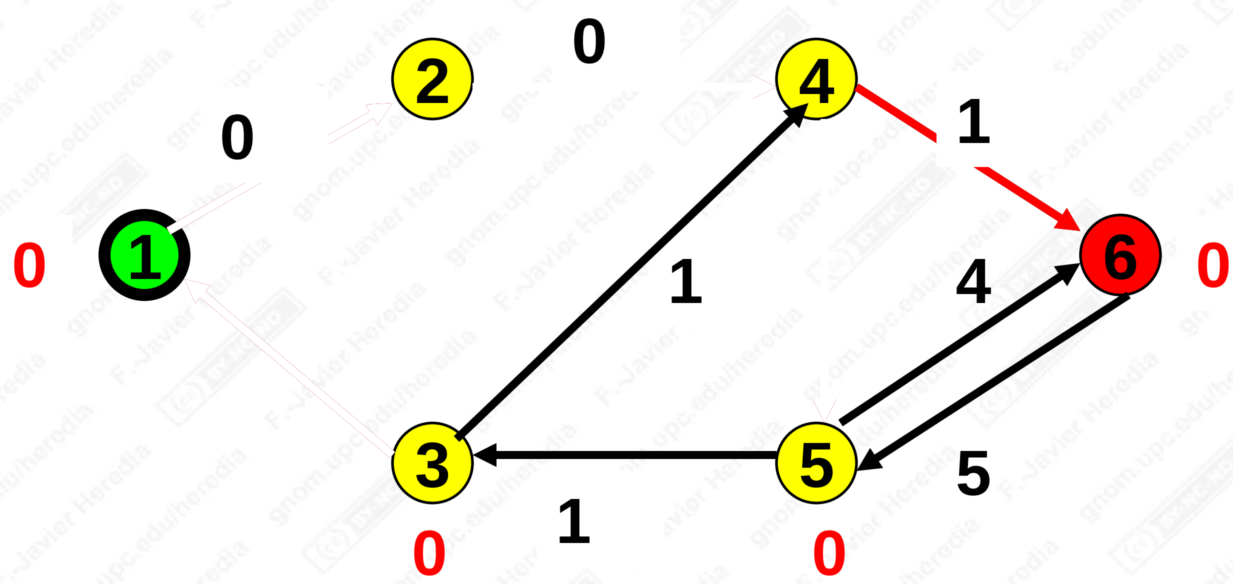
Update

update the current flow and decomposition



cycle flows

2 units around 1-2-4-5-3-1

3 units around 2-4-5-3-2

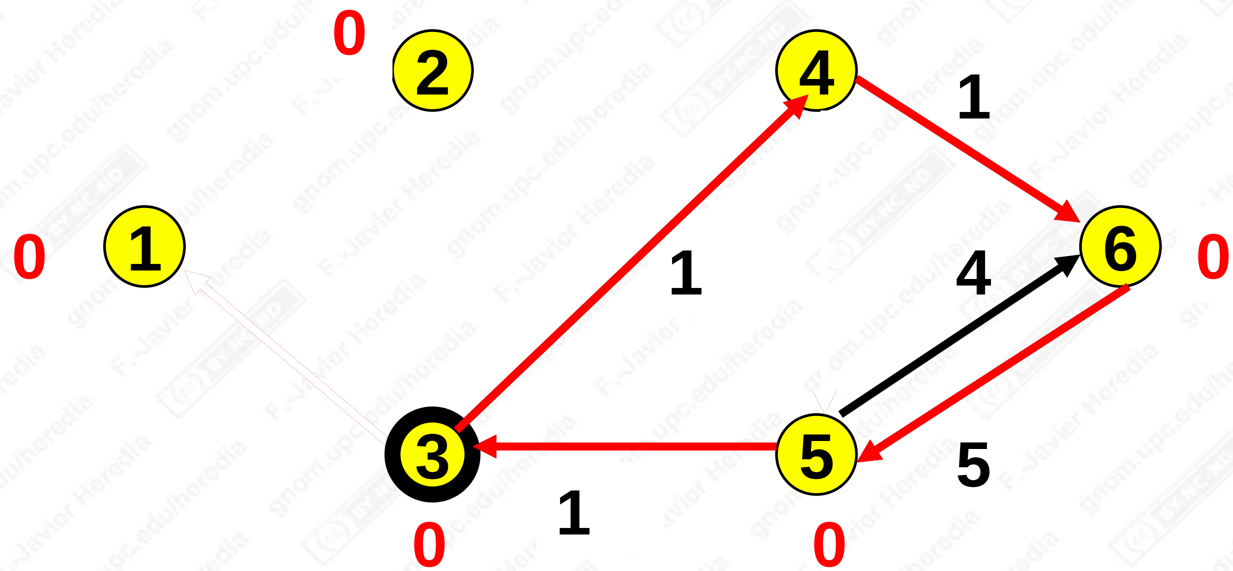
3 units around 4-5-3-4

path flows

2 units in 1-2-4-6

Find the next path or cycle

start with any node incident to an arc with flow and find the next flow or cycle.



cycle flows

2 units around 1-2-4-5-3-1

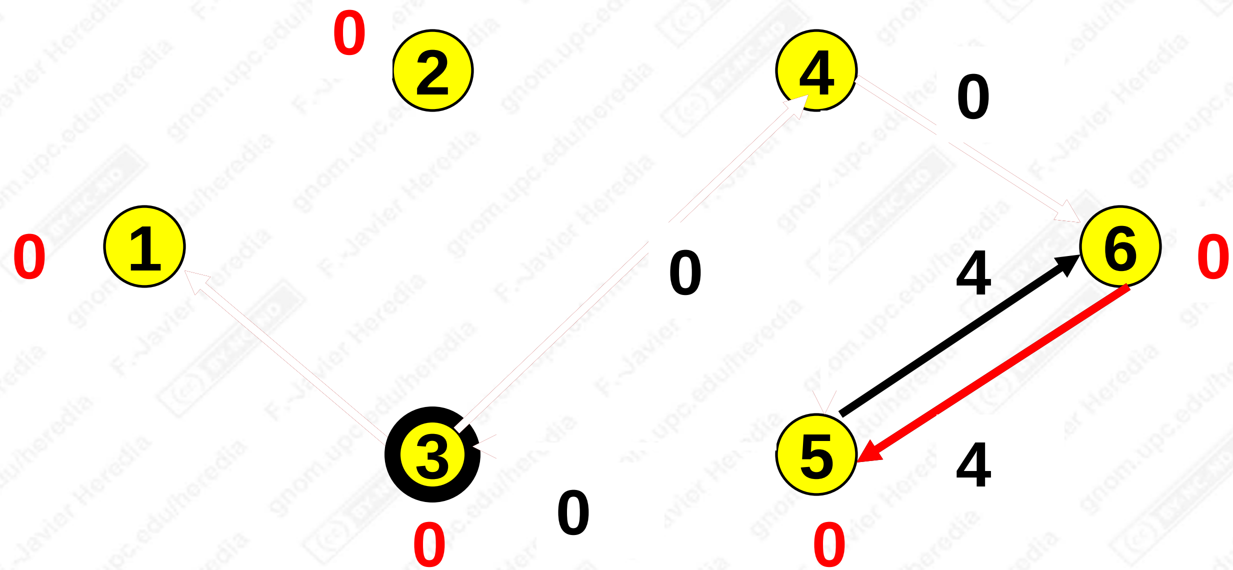
3 units around 2-4-5-3-2

3 units around 4-5-3-4

path flows

2 units in 1-2-4-6

Update



cycle flows

2 units around 1-2-4-5-3-1

3 units around 2-4-5-3-2

3 units around 4-5-3-4

1 units around 3-4-6-5-3

path flows

2 units in 1-2-4-6

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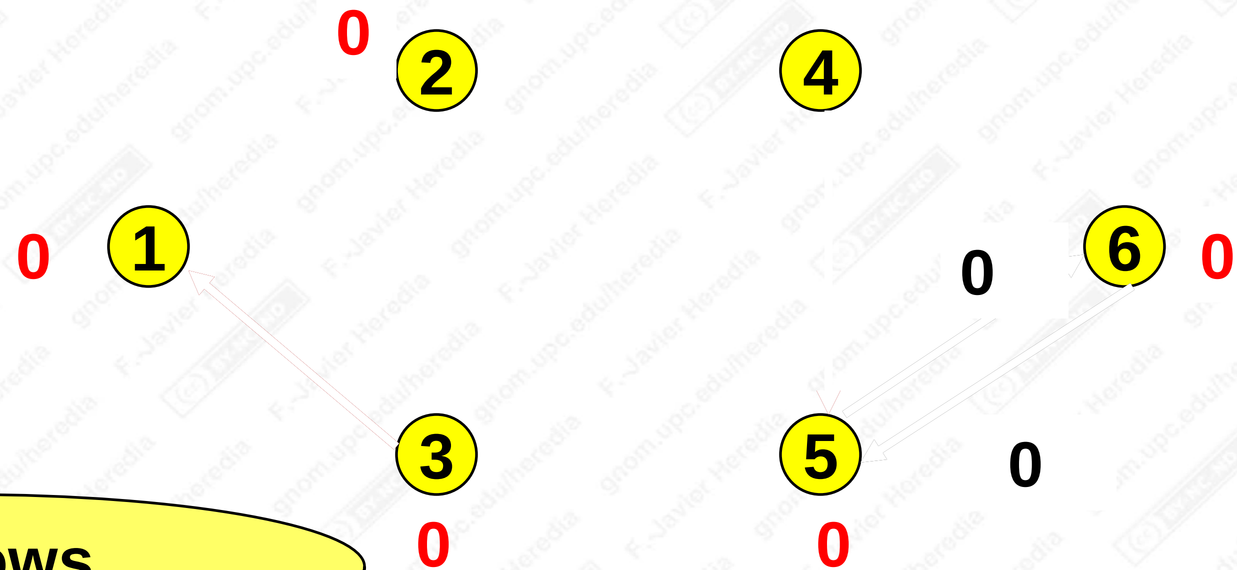
The Final Decomposition



cycle flows

2 units around 1-2-4-5-3-1

3 units around 2-4-5-3-2

3 units around 4-5-3-4

1 units around 3-4-6-5-3

4 units around 5-6-5

path flows

2 units in 1-2-4-6

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Column Generation: Arc-Path Formulation

- **O-D multicommodity flow problems**: Let s_k be the unique source of commodity k and t_k be the unique sink of commodity k.
- **Additional assumption**: for every commodity, the cost of every cycle W in the underlying network is nonnegative

$\Rightarrow$ for some optimal solution, the flow on every cycle is zero

$\Rightarrow$ any optimal solution can be represented as the sum of flows on directed paths

This is the case if $c_{ij}^k \geq 0 \ \forall$ i,j,k.

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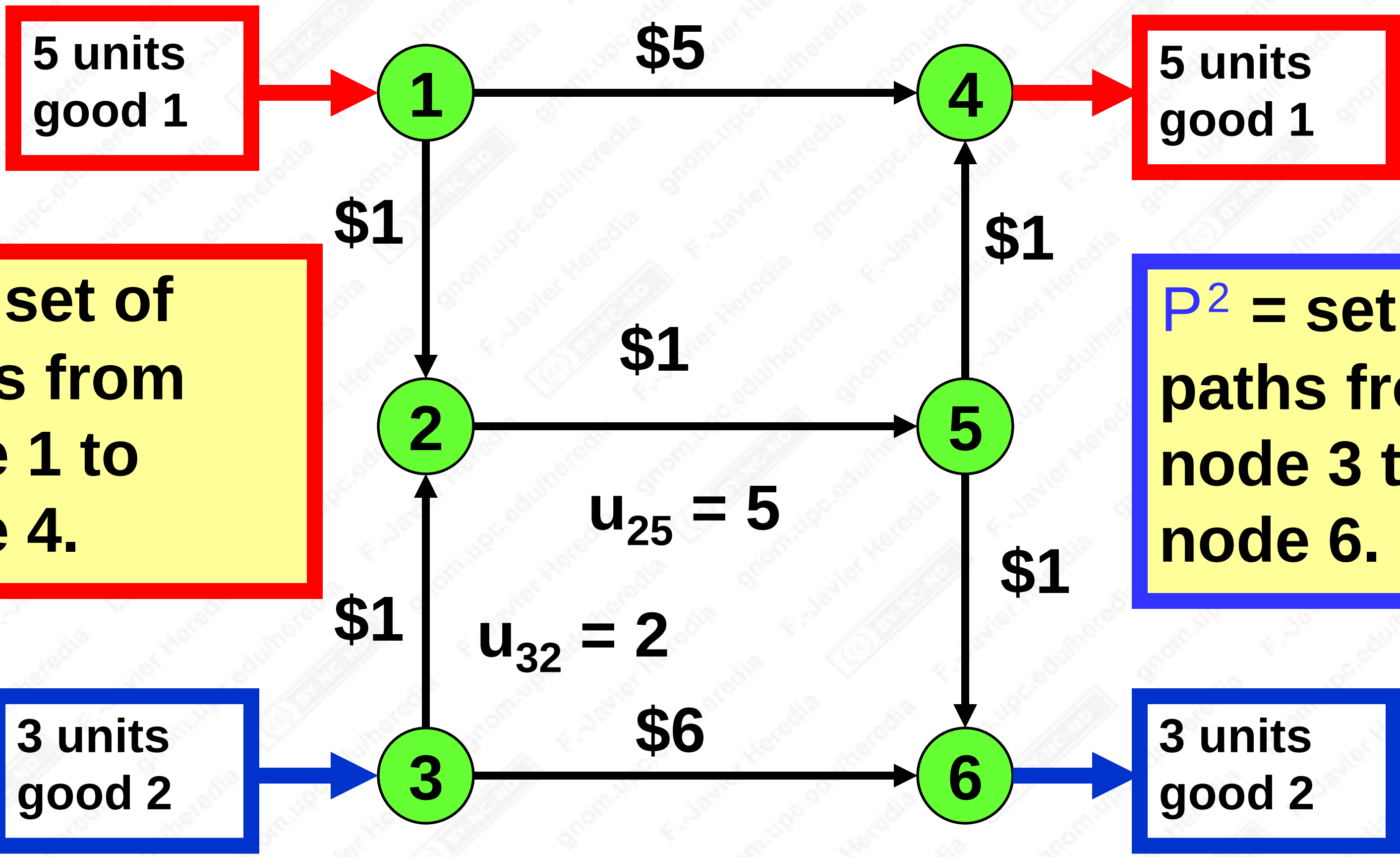
A Linear Multicommodity Flow Problem



P^1 = set of paths from node 1 to node 4.

P^2 = set of paths from node 3 to node 6.

P^1 = {1-4, 1-2-5-4}

P^2 = {3-6, 3-2-5-6}

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A path based formulation

f(P) = flow in path P

c(P) = cost of path P

c(1-4)	=	5
c(1-2-5-4)	=	3
c(3-6)	=	6
c(3-2-5-6)	=	3

Minimize 5 f(1-4) + 3 f(1-2-5-4) + 6 f(3-6) + 3 f(3-2-5-6)

subject to

f(1-4) + f(1-2-5-4) = 5

f(3-6) + f(3-2-5-6) = 3

f(1-2-5-4) + f(3-2-5-6) $\leq$ $u_{25} = 5$

f(3-2-5-6) $\leq$ $u_{32} = 2$

f(P) $\geq$ 0 for all paths P

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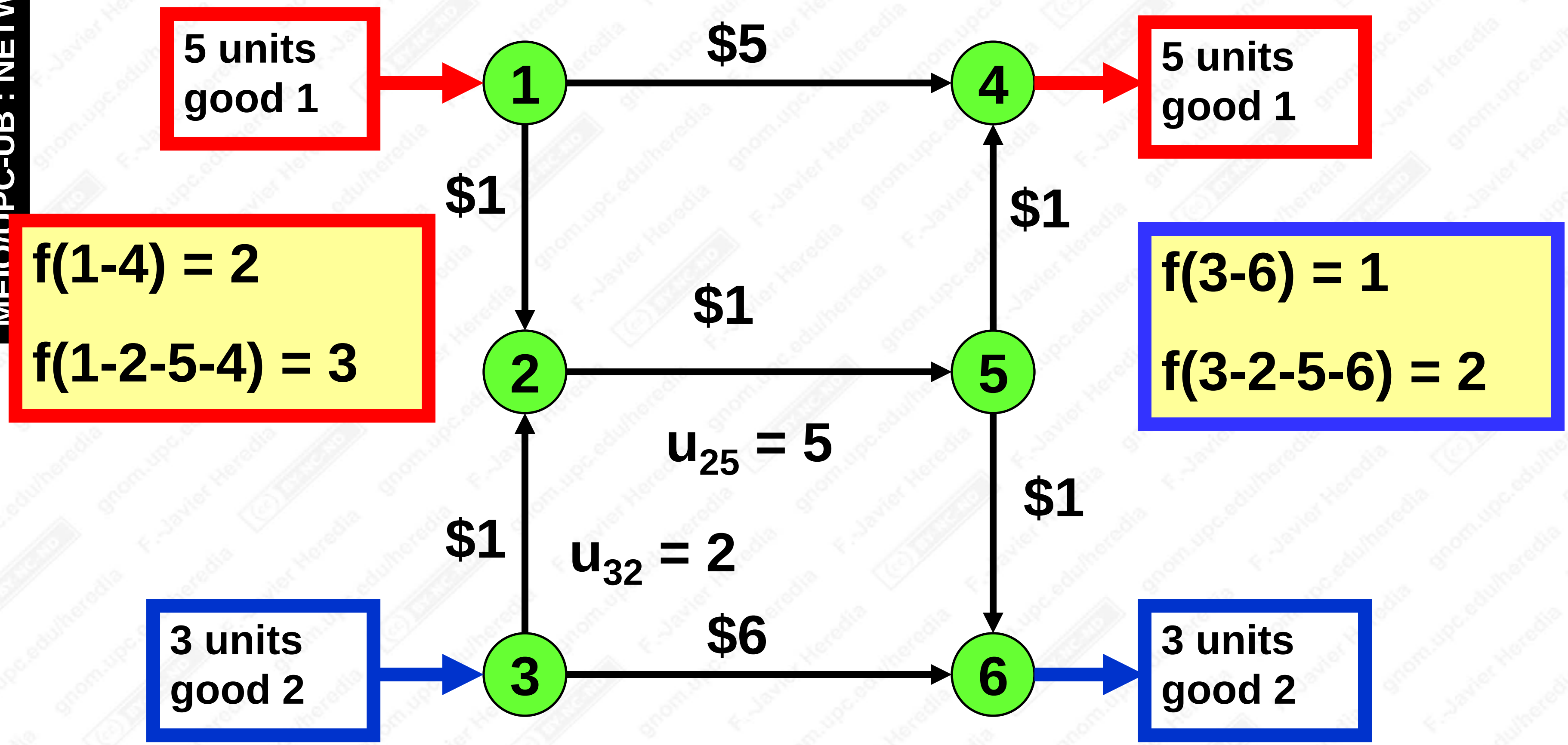
Optimal solution for the path based version



The path based LP can be solved using the simplex method.

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General formulation for the path based version

Let P^k = set of directed paths from s_k to t_k

Let $c^k(P)$ = cost of path $P \in P^k$.

Let $f(P)$ = flow on path P.

$$\text{Let } \delta_{ij}(P) = \begin{cases} 1 & \text{if } (i,j) \in P \\ 0 & \text{otherwise} \end{cases}$$

Master Problem

$$\text{Minimize} \quad \sum_k \sum_{P \in P^k} c^k(P) f(P)$$

$$\sum_k \sum_{P \in P^k} \delta_{ij}(P) f(P) \le u_{ij} \qquad \text{for all } (i,j) \in A$$

$$\sum_{P \in P^k} f(P) = d^k \qquad \text{for } k = 1 \text{ to } K$$

$$f(P) \ge 0 \qquad \text{for } P \in \bigcup_{k=1}^{K} P^k$$

Remarks on the path based version

$$\text{Minimize} \quad \sum_{k} \sum_{P \in P^k} c^k(P) f(P)$$

$$\sum_{k} \sum_{P \in P^k} \delta_{ij}(P) f(P) \le u_{ij} \qquad \text{for all } (i,j) \in A$$

$$\sum_{P \in P^k} f(P) = d^k \qquad \text{for } k = 1 \text{ to } K$$

$$f(P) \ge 0 \qquad \text{for } P \in \bigcup_{k=1}^{K} P^k$$

- **Contraints:** (m+K << m+nK node-arc formulation)

  Bundle constraints: one for each capacitated arc (m).

  Supply/demand constraints: one for commodity (K).

- **Variables**: one for each path from origin to destination for each commodity : exponential!!

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Column Generation Approach: generate paths as needed

Let S^k = subset of P^k = directed paths from s_k to t_k

Let $c^k(P)$ = cost of path $P \in S^k$.

Let $f(P)$ = flow on path P.

$$\text{Let } \delta_{ij}(P) = \begin{cases} 1 & \text{if } (i,j) \in P \\ 0 & \text{otherwise} \end{cases}$$

Restricted Master Problem

$$\text{Minimize} \quad \sum_k \sum_{P \in S^k} c^k(P) f(P)$$

$$\sum_k \sum_{P \in S^k} \delta_{ij}(P) f(P) \le u_{ij} \qquad \text{for all } (i,j) \in A$$

$$\sum_{P \in S^k} f(P) = d^k \qquad \text{for } k = 1 \text{ to } K$$

$$f(P) \ge 0 \qquad \text{for } P \in \bigcup_{k=1}^{K} S^k$$

Solving the Master Problem

1. **Initialize S^k for each k.**
2. **Apply LP to solve the restricted master problem for paths in $S = \cup_k S^k$ obtaining solution f(P).**
3. **Check to see if f(P) is optimal for the master problem. If not, find new paths to add to S and return to step 2.**

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Towards Optimality conditions for the master and restricted master

Restricted Master Problem

$$\text{Minimize} \quad \sum_k \sum_{P \in S^k} c^k(P) f(P)$$

$$\sum_k \sum_{P \in S^k} \delta_{ij}(P) f(P) \le u_{ij} \quad \text{for all } (i,j) \in A$$

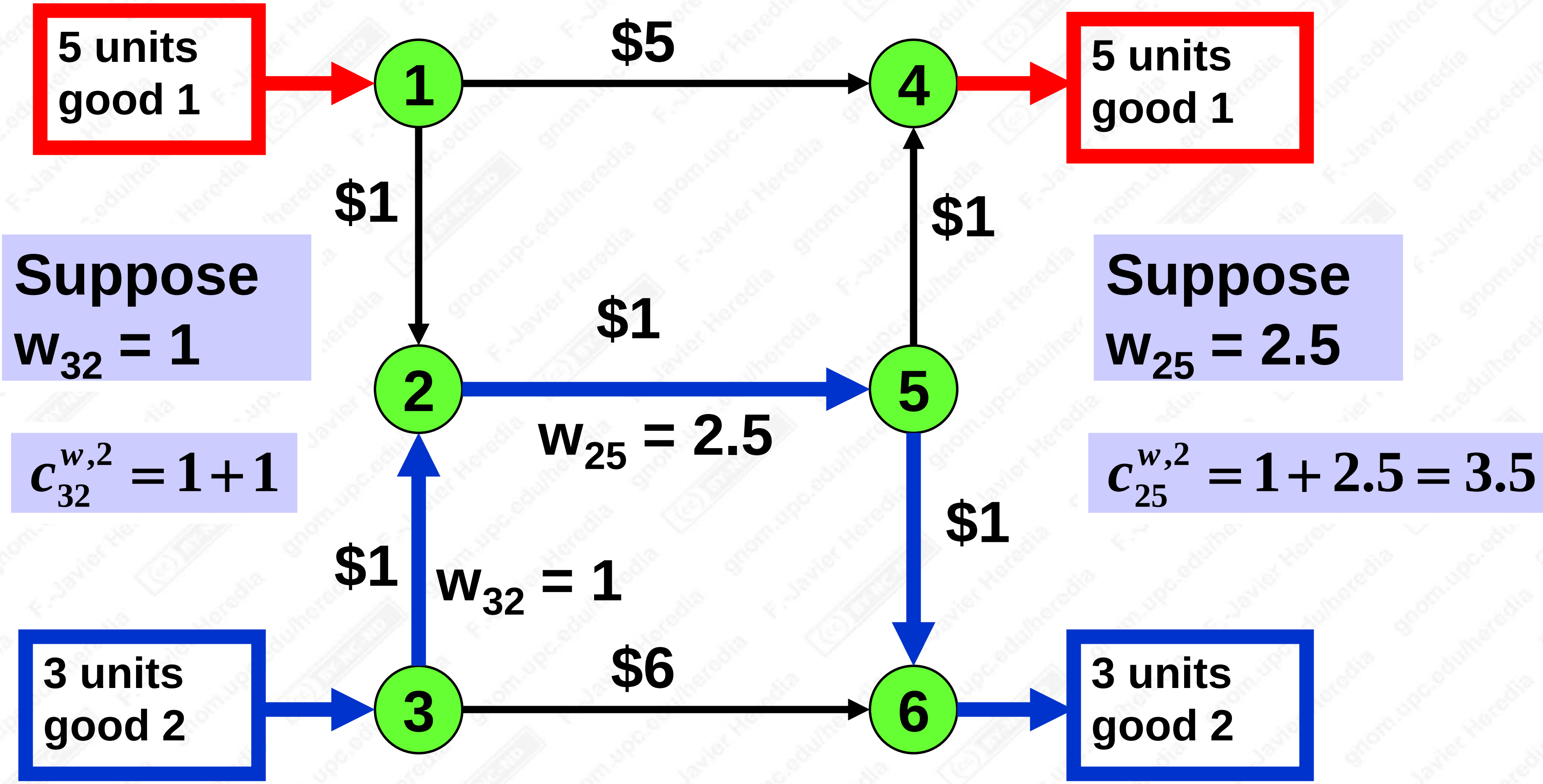
$$\sum_{P \in S^k} f(P) = d^k$$

$$f(P) \ge 0 \quad \text{for } P \in \bigcup_{k=1}^{K} S^k$$

Let w = (w_{ij}) be a set of non-negative tolls on the arcs.

$$c_{ij}^{w,k} = c_{ij}^k + w_{ij} \qquad c^{w,k}(P) = \sum_{(i,j)\in P} c_{ij}^{w,k}$$

Illustration of definitions



$$c^{w,2}(3-2-5-6) = 2 + 3.5 + 1 = 6.5$$

Optimality conditions for the restricted master

A flow f is optimal for the restricted master if it is feasible, and if there is a non-negative vector w of tolls on arc so that the following is true:

1. $w_{ij} > 0 \Rightarrow \sum_{k} \sum_{P \in S^k} \delta_{ij}(P) f(P) = u_{ij}$ **for all (i,j)** $\in$ **A**

2. $f(P') > 0$ **for P'** $\in$ $\mathbf{S^k}$

   $\Rightarrow \mathbf{c}^{w,k}(P')$ $= \min (c^{w,k}(P) : P \in S^k)$

The tolls w are produced by the LP solution to the restricted master (shadow prices of the bundle constraints).

Optimality for the master problem is the same except that S^k is replaced by P^k.

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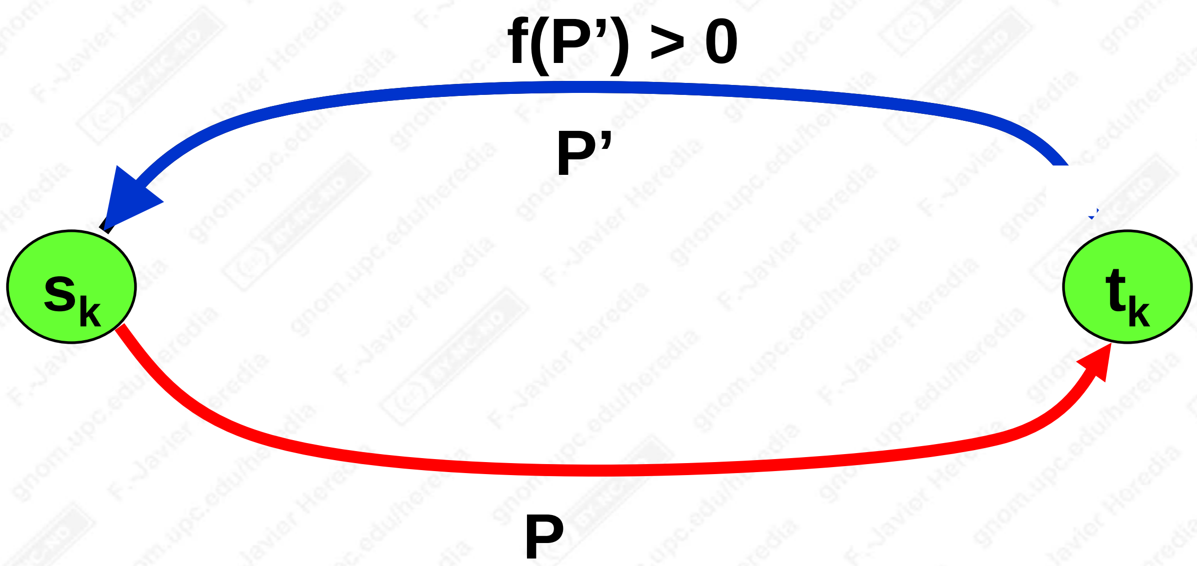
On the optimality conditions



Suppose that the optimality conditions are not satisfied, and that c(P’) > c(P).

The residual network $G(x^k)$ has Rev(P’).

Rev(P’) + P is a circulation with negative cost -c(P’) +c(P) < 0.

Thus $G(x^k)$ has a negative cost cycle.

Adding paths or proving optimality

Let **f** be optimal for the restricted master

1. $$w_{ij} > 0 \Rightarrow \sum_{k} \sum_{P \in S^k} \delta_{ij}(P) f(P) = u_{ij} \text{ for all } (i,j) \in A$$

2. $$f(P') > 0 \text{ for } P' \in S^k$$

$$\Rightarrow c^{w,k}(P') = \min\left(c^{w,k}(P) : P \in S^k\right)$$

How can be checked if **f** is **optimal for the master problem**:

1. Let P^k be the shortest path from s_k to t_k using $c^{w,k}$.
2. If $P^k \in S^k$ for each k, then **f** is optimal for the master problem because condition 1 and condition 2 are both satisfied when we replace S^k by P^k for each k.
3. Otherwise, add P^k to S^k for each k, and solve the new restricted master problem.

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More on optimality conditions

- We used partial dualization. That is, we had optimality conditions that used w, but not σ^k the dual variables of the supply/demand constraints of commodity *k*.
- In usual column generation, one would use LP **path flow complementarity slackness optimality conditions**, which are more detailed, but follow the same general approach (See A-M-H, section 17.5).

(a) $w_{ij}\left[\sum_{1\le k\le K}\sum_{P\in \mathrm{P}^k}\delta_{ij}(P)f(P)-u_{ij}\right]=0 \quad \forall (i,j)\in A$

(b) $c^{\sigma,w}(P)\ge 0 \quad \forall k=1,2,\ldots,K\ ,\forall P\in \mathrm{P}^k$

(c) $c^{\sigma,w}(P)f(P)=0 \quad \forall k=1,2,\ldots,K\ ,\forall P\in \mathrm{P}^k$

with $c^{\sigma,w}(P)=\sum_{(i,j)\in P}\left(c_{ij}^k+w_{ij}\right)-\sigma^k$ the reduced cost of variable $f(P)$ and σ^k the dual variable of comm. *k*, which can also be interpreted as the shortest path distance from s_k to t_k w.r.t. the modified costs $c_{ij}^k+w_{ij}$

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Restricted Master Problem 1

f(P) = flow in path P

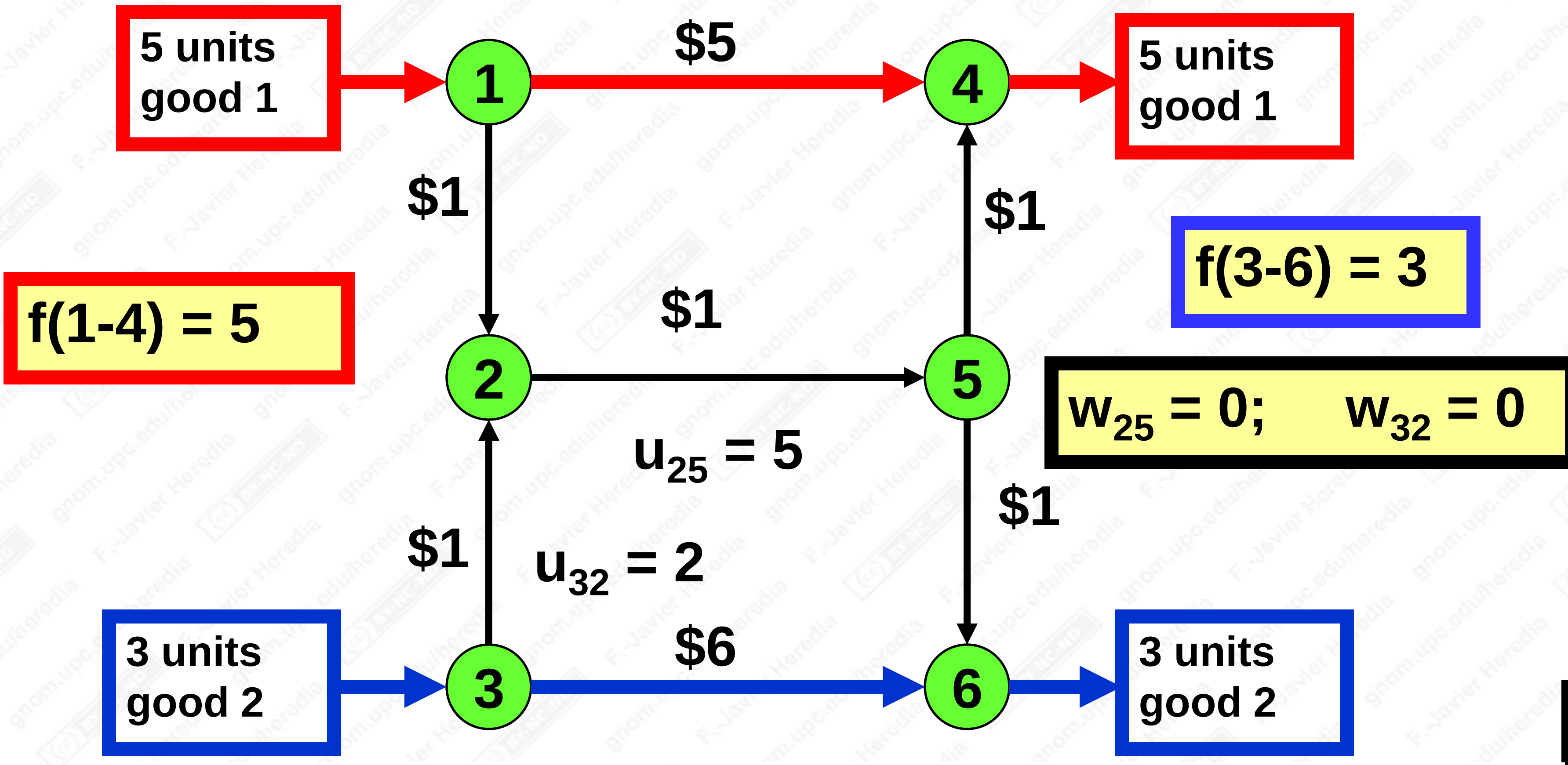
c(P) = cost of path P

c(1-4)	=	5
c(3-6)	=	6

Minimize	5 f(1-4)	6 f(3-6)		
subject to	f(1-4)		=	5
	f(3-6)		=	3

f(P) ≥ 0 for all paths P

Optimal solution for restricted master 1



The unique shortest path for commodity 1 is 1-2-5-4.

The unique shortest path for commodity 2 is 3-2-5-6.

Restricted Master Problem 2

Suppose we add path 3-2-5-6 to the restricted master

f(P) = flow in path P

c(P) = cost of path P

c(1-4)	=	5
c(3-6)	=	6
c(3-2-5-6)	=	3

$$\text{Minimize} \quad 5\,f(1\text{-}4) \quad 6\,f(3\text{-}6) + 3\,f(3\text{-}2\text{-}5\text{-}6)$$

$$\text{subject to} \quad f(1\text{-}4) = 5$$

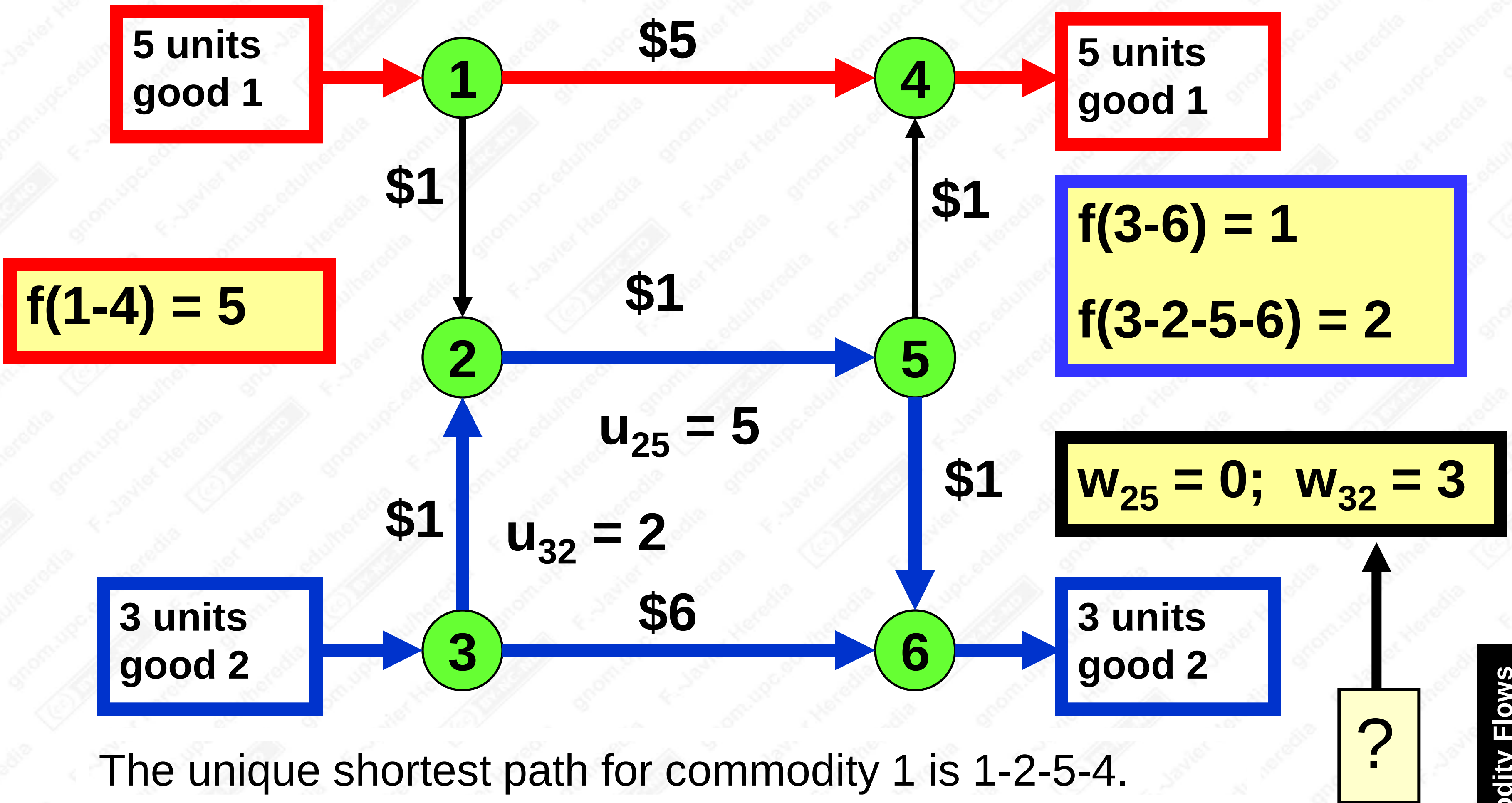
$$f(3\text{-}6) + f(3\text{-}2\text{-}5\text{-}6) = 3$$

$$f(3\text{-}2\text{-}5\text{-}6) \leq u_{25} = 5$$

$$f(3\text{-}2\text{-}5\text{-}6) \leq u_{32} = 2$$

$$f(P) \geq 0 \text{ for all paths } P$$

Optimal solution for restricted master 2



The unique shortest path for commodity 1 is 1-2-5-4.

The shortest paths for commodity 2 are 3-2-5-6 and 3-6

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Restricted Master Problem 3

We next add path 1-2-5-4 to the restricted master

f(P) = flow in path P

c(P) = cost of path P

c(1-4)	=	5
c(1-2-5-4)	=	3
c(3-6)	=	6
c(3-2-5-6)	=	3

Minimize 5 f(1-4) + 3 f(1-2-5-4) + 6 f(3-6) + 3 f(3-2-5-6)

subject to

$$f(1\text{-}4) + f(1\text{-}2\text{-}5\text{-}4) = 5$$

$$f(3\text{-}6) + f(3\text{-}2\text{-}5\text{-}6) = 3$$

$$f(1\text{-}2\text{-}5\text{-}4) + f(3\text{-}2\text{-}5\text{-}6) \le u_{25} = 5$$

$$f(3\text{-}2\text{-}5\text{-}6) \le u_{32} = 2$$

$$f(P) \ge 0 \text{ for all paths } P$$

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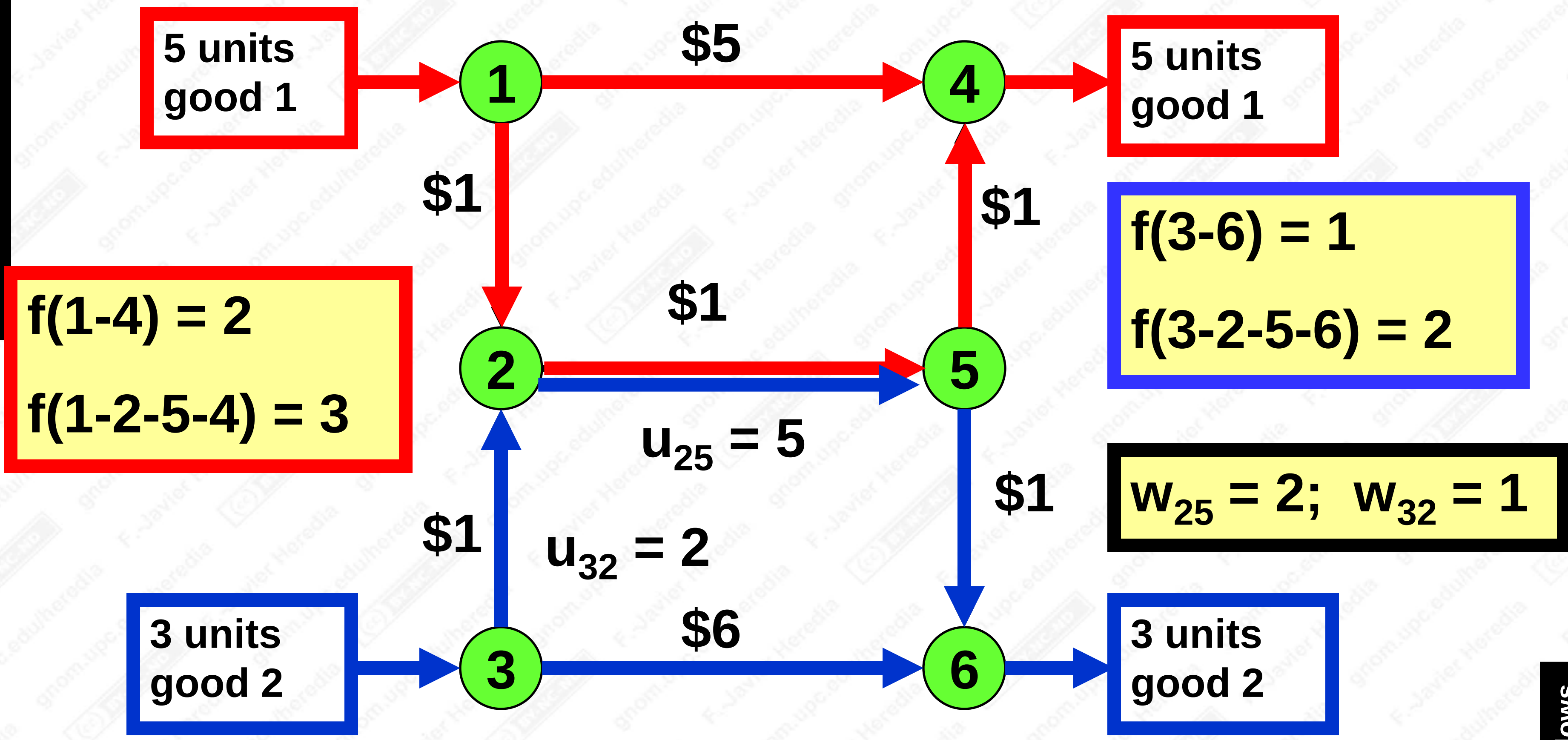
Optimal solution for the path based version



The solution is optimal for the entire problem.

Column Generation

Restricted Master Problem (RMP)

> Trillions (?) of Variables

Constraints

Initial variables

Added variables

Variables that were never considered

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