Excerpt from an exam

Consider $\mathcal{P}_2$ vector space of polynomials from $\mathbb{R} \to \mathbb{R}$ whose degree is ≤ 2. Consider the polynomials e_1, e_2, $e_3 \in \mathcal{P}_2$ defined by the following $\forall x \in \mathbb{R}$ we have

$$e_1(x) = 3 - x$$
$$e_2(x) = x^2 - x + 1$$
$$e_3(x) = x^2 - 3x + 2.$$

Are e_1, e_2, e_3 linearly independant.

Answer

The dimension of $\mathcal{P}_n$ is $n+1$, thus $\mathcal{P}_2$ has dimension 3. The standard basis of $\mathcal{P}_2$ is composed with polynomials $P_0, P_1, P_2 \in \mathcal{P}_2$ such that $\forall x \in \mathbb{R}$ we have

$p_0(x) = 1$

$p_1(x) = x$

$p_2(x) = x^2$

Thus $e_1 = 3p_0 - p_1$

$e_2 = p_2 - p_1 + p_0$

$e_3 = p_2 - 3p_1 + 2p_0$

So that

$e_1 = (3, -1, 0)$

$e_2 = (1, -1, 1)$

$e_3 = (2, -3, 1)$

Compute $\begin{vmatrix} 3 & 1 & 2 \\ -1 & -1 & -3 \\ 0 & 1 & 1 \end{vmatrix} = 5 \neq 0$

Since $\dim \mathcal{P}_2 = 3$ it follows that e_1, e_2, e_3 is a basis of $\mathcal{P}_2$ which means that they are linearly independant.

A student's answer

$e_1, e_2, e_3 \in \mathcal{P}_2$

$\forall \lambda_1, \lambda_2, \lambda_3 \rightarrow \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 = 0_V$

$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$

Our objective is to show that

$\forall \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, $\cancel{x}$

$$\Rightarrow 0 \cdot (3 - x) + 0 \cdot (x^2 - x + 1) + 0 \cdot (x^2 - 3x + 2) =$$

$$0 \cdot 3 - 0 \cdot x + 0x^2 - 0x + 0 \cdot 1 + 0x^2 - 3 \times 0 + 0 \cdot 2 = 0$$

QED are linearly independent

Student's answer

$$\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 = \vec{0} \quad (1)$$

$\boxed{\text{Let } \lambda_1, \lambda_2, \lambda_3 \text{ be reals numbers.}}$

$$\lambda_1(3-x) + \lambda_2(x^2-x+1) + \lambda_3(x^2-3x+2) = 0$$

$$\boxed{(1) \iff \forall x \in \mathbb{R}, \quad \lambda_1(3-x) + \lambda_2(x^2-x+1) + \lambda_3(x^2-3x+2) = 0 \quad (2)}$$

$$\lambda_1(3-x) = 0$$
$$\lambda_2(x^2-x+1) = 0$$
$$\lambda_3(x^2-3x+2) = 0$$

$\longrightarrow$ INCORRECT

Student's answer

$$\lambda_1(3-x) + \lambda_2(x^2-x+1) + \lambda_3(x^2-3x+2) = 0$$

$\rightarrow$ thus $\lambda_1 = \lambda_2 = \lambda_3 = 0$

thus e_1, e_2, e_3 are linearly independant