Exercise (excerpt from an exam)

We consider the vector space $(\mathbb{R}^2, +, \cdot)$. Each vector $(x,y) \neq (0,0)$ is written in polar coordinates as

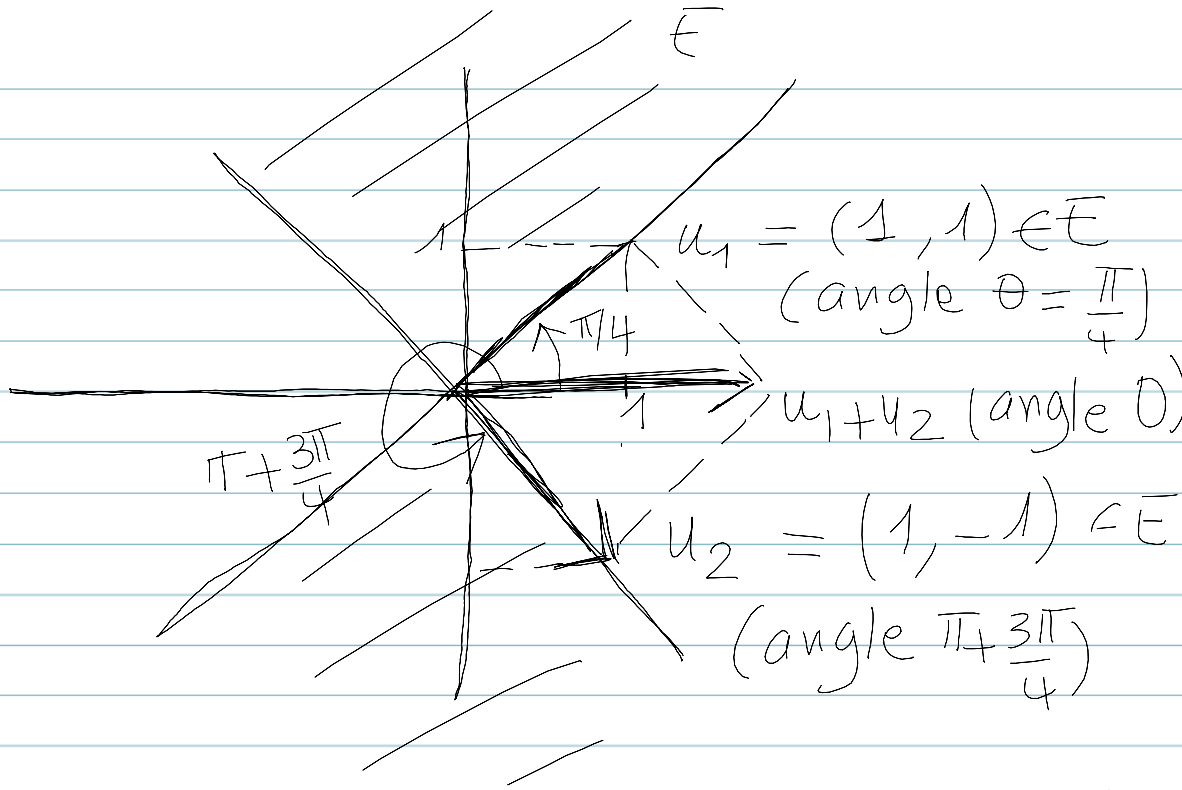
$$x = r\cos(\theta)$$

$$y = r\sin(\theta) \quad \text{where } r > 0$$

and $\theta \in [0, 2\pi)$. Consider the set $E = \{(0,0)\} \cup$

$$\left\{ (r\cos(\theta), r\sin(\theta)) \in \mathbb{R}^2 \,/\, r > 0 \text{ and } \theta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right] \cup \left[\pi + \frac{\pi}{4}, \pi + \frac{3\pi}{4}\right] \right\}$$

Is E a subspace of $\mathbb{R}^2$?



$$u_1 + u_2 = (1,1) + (1,-1) = (2,0) \notin E$$

Thus we have found $u_1 \in E$, $u_2 \in E$ such that $u_1 + u_2 \notin E$

This means that E is NOT a subspace of $\mathbb{R}^2$

<u>Another method</u>:

We know that the only subspaces of $\mathbb{R}^2$ are: $\{(0,0)\}$, the lines that go through $(0,0)$ and $\mathbb{R}^2$. Since E is none of these it cannot be a subspace of $\mathbb{R}^2$.

Student's answer

If we want E to be a subspace of $\mathbb{R}^2$, it has to follow this

$$\begin{cases} \forall x,y \in E, \Rightarrow (x+y) \in E & (1) \\ \forall x \in E, \forall \lambda \in \mathbb{R} \Rightarrow \lambda x \in E & (2) \end{cases}$$

INCORRECT

$\forall x,y \in E$ we have $x+y \in E$

$\forall x \in E, \forall \lambda \in \mathbb{R}$ we have $\lambda x \in E$

We will consider

$\vec{v} = (r_\xi \cos\theta, r_\xi \sin\theta)$

$\vec{u} : (r_\xi \sin\theta, r_\xi \cos\theta)$

?

Condition 1

$\vec{v} + \vec{u} = (r_\xi \cos\theta + r_\xi \sin\theta, r_\xi \sin\theta + r_\xi \cos\theta)$

Correct

$\forall x, y \in E$

arbitrary

$\cos(\theta)$

$\cancel{\cos\theta + r\sin\theta}$

$$\vec{v} + \vec{u} = \left(r(\cos\theta + \sin\theta), r(\sin\theta + \cos\theta)\right)$$

→ We cannot prove that it is in E.

INCORRECT

Condition 2

$$\alpha\vec{v} = (\alpha r\cos\theta, \alpha r\sin\theta)$$

CORRECT: let's consider some $\alpha \in \mathbb{R}$ and ~~wirte~~ write $\vec{v}$ in polar coordinates as $\vec{v} = (r\cos(\theta), r\sin(\theta))$

We can prove that it is not a subspace with numbers by showing that 2 is not satisfied.

Let's take angle $\frac{\pi}{2} = \theta$
and $\alpha = -3$

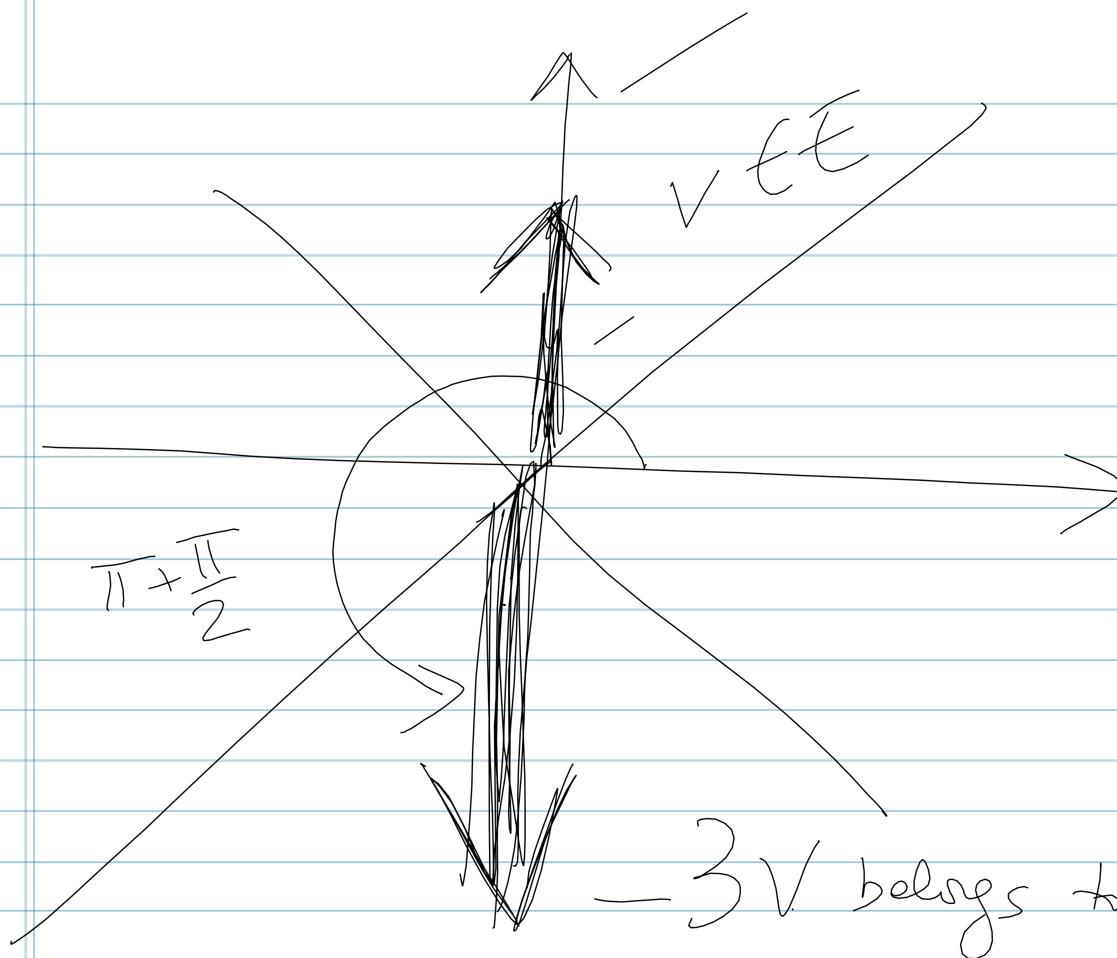
$$-3v = \left(-3r\cos\frac{\pi}{2}, -3r\sin\frac{\pi}{2}\right)$$

$\alpha \in \mathbb{R}$ → so it can be negative
$r > 0$

So $\alpha \cdot v \notin E$.

is < 0 $\Rightarrow$ there is a problem

$$\boxed{-3r\sin\frac{\pi}{2} = \underline{\underline{3r}}\sin\left(\frac{\pi}{2}+\pi\right)}$$



The argument of the student that $\alpha v \notin E$ IS INCORRECT

Conclusion of the student:

E is not a subspace.