Excerpt from an exam

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be such that $\forall (x,y) \in \mathbb{R}^2$ we have

$$f(x,y) = \frac{\sin(x^2)}{2} - \frac{y^2}{2}$$

Find the local extrema of f.

<u>Answer</u>: The function f is infinitely differentiable as a combination of polynomials and a sine function. ~~Then~~ we can use the method for finding local extrema.

To find the local extrema, we solve

$$\frac{\partial f}{\partial x}(x,y) = x\cos(x^2) = 0 \quad (1)$$

and

$$\frac{\partial f}{\partial y}(x,y) = -y = 0 \quad (2)$$

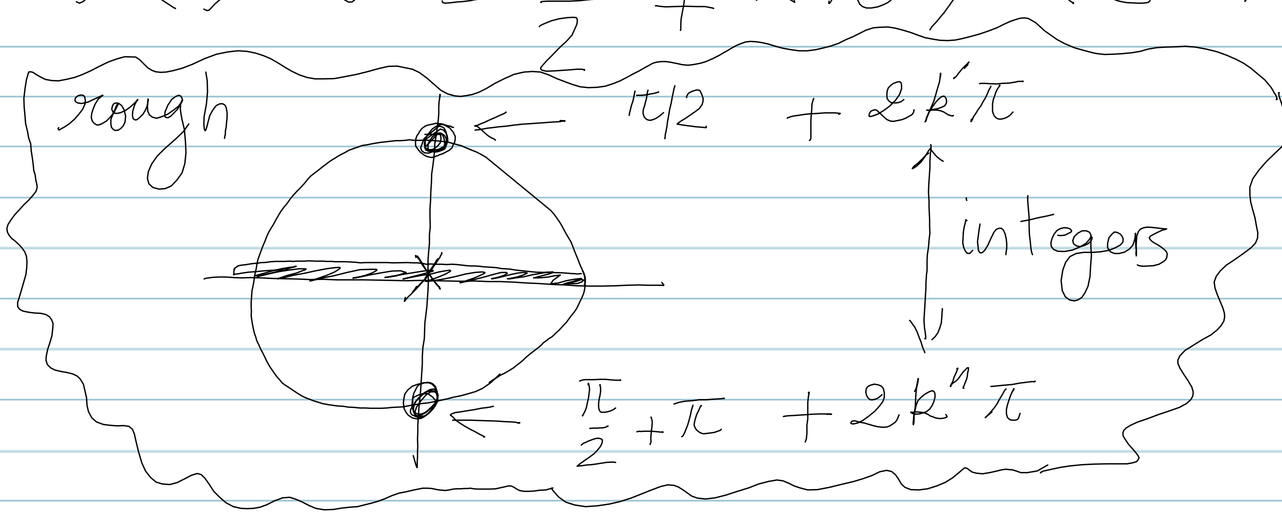
Equation (1) gives $x = 0$ or $\cos(x^2) = 0$.

Equation (2) gives $y = 0$

$\Rightarrow$ We get a critical point $(0,0)$ and we now find the solutions of

$$\cos(x^2) = 0 \quad (3)$$

$$(3) \iff x^2 = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{N}$$



k cannot be negative $(-1, -2, \text{etc})$ because $x^2 \geq 0$.

This gives a set of critical points

$$\left(\pm\sqrt{\frac{\pi}{2} + k\pi}\,,\ 0\right)$$

$$k \in \mathbb{N}$$

The Hessian matrix is given by

$$H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} \cos(x^2) - 2x^2\sin(x^2) & 0 \\ 0 & -1 \end{pmatrix}$$

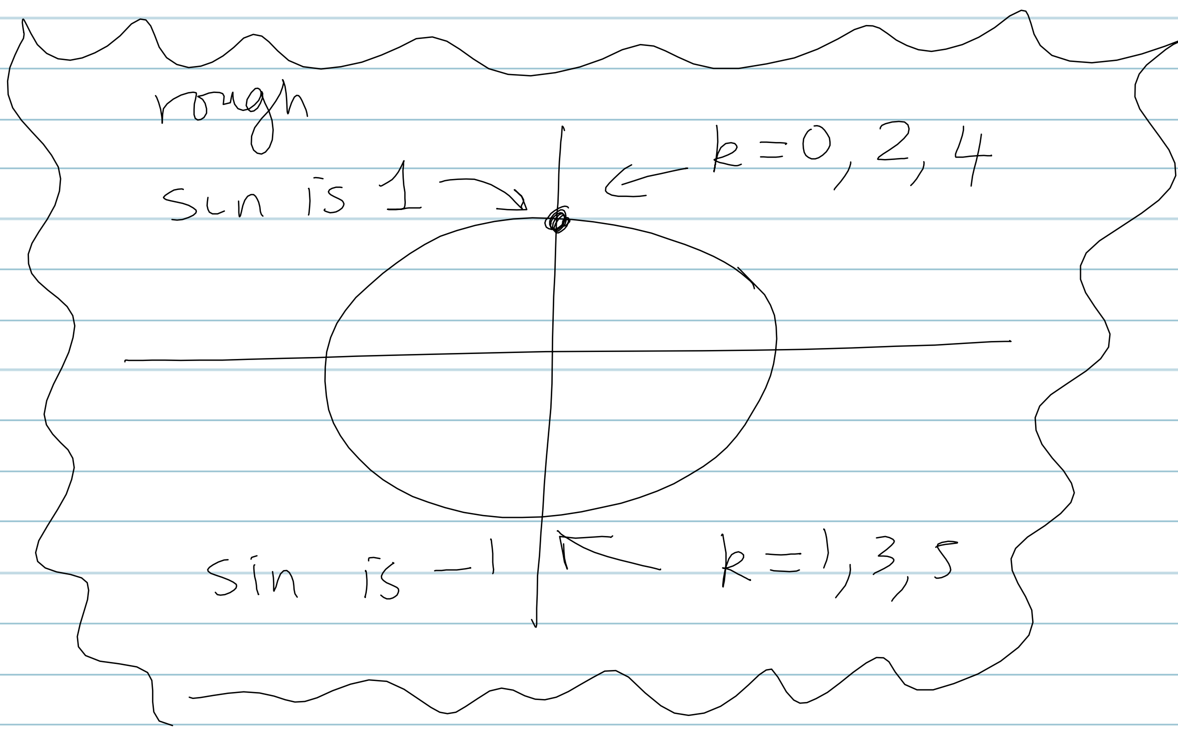
$$H_f(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ so that

$\det\left(H_f(0,0)\right) = -1 < 0$. Thus $(0,0)$ is a saddle point.

Evaluate the matrix $H_f(x,y)$ at the critical points $\left(\pm\sqrt{\frac{\pi}{2}+k\pi},\ 0\right)$

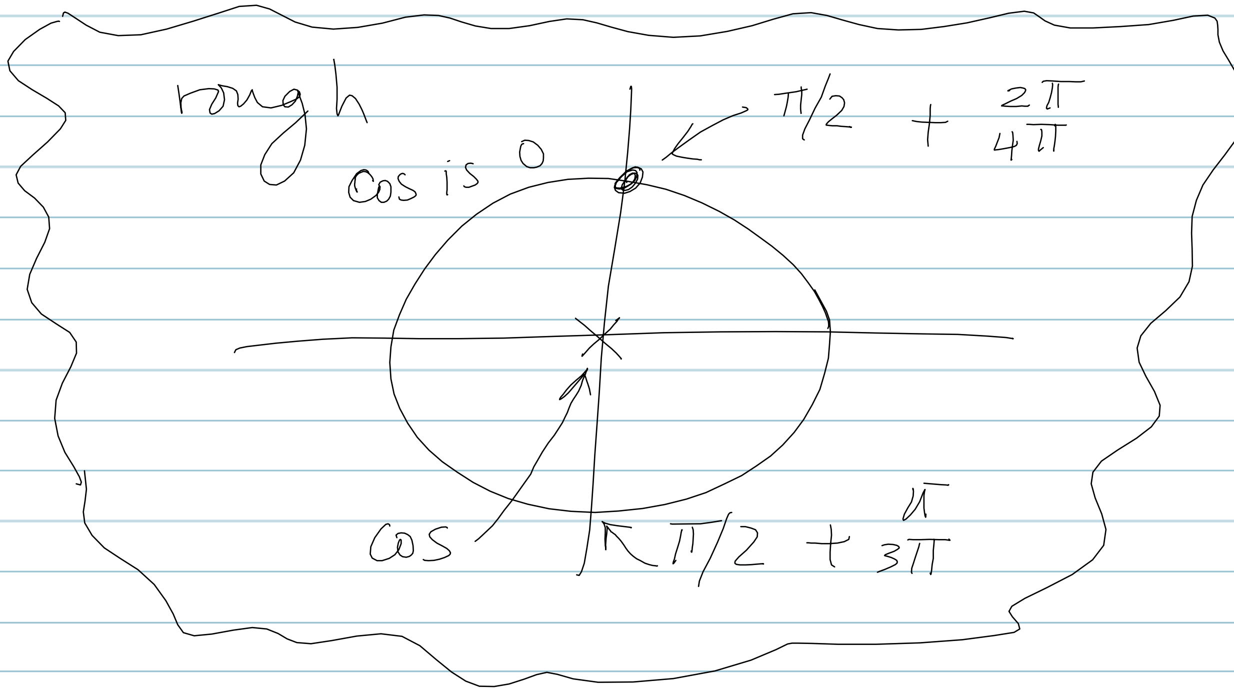
$k \in \mathbb{N}$

$$\sin(x^2) = \sin\left(\frac{\pi}{2} + k\pi\right)$$



$$= (-1)^k$$

$$\cos(x^2) = \cos\left(\frac{\pi}{2} + k\pi\right)$$



$$= 0$$

Thus

$$\cos(x^2) - 2x^2 \sin(x^2) =$$

$$-2\left(\frac{\pi}{2} + k\pi\right) \cdot (-1)^k$$

$$\det\left(H_f\left(\pm\sqrt{\frac{\pi}{2} + k\pi},\ 0\right)\right)$$

$$= 2\underbrace{\left(\frac{\pi}{2} + k\pi\right)}_{>0 \text{ because } k \in \mathbb{N}} \cdot (-1)^k$$

We discuss the following cases

k is odd, $(-1)^k = -1$ so that $\det(H_f) < 0$. The critical point $\left(\pm\sqrt{\frac{\pi}{2}+k\pi}, 0\right)$ corresponds to a saddle point.

k is even, $(-1)^k = 1$ so that $\det(H_f) > 0$.

$$\frac{\partial^2 f}{\partial x^2}(x,y) = -2\underbrace{\left(\frac{\pi}{2}+k\pi\right)}_{>0} < 0$$

This implies that the critical points $\left(\pm\sqrt{\frac{\pi}{2}+k\pi}, 0\right)$ correspond to a local minimum.

Student's answer

$$f' = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (0,0)$$

$$\Rightarrow \begin{cases} \dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} = 0 \end{cases}$$

$$\frac{\partial f}{\partial x} = \cos(x^2), \ x = 0 \begin{cases} x = 0 \\ x^2 = \dfrac{\pi}{2} \end{cases}$$

?

anything missing ?

$$\rightarrow x = \underbrace{\pm\sqrt{\frac{\pi}{2} + \pi k}}_{\text{mistake}}, \quad k \in \mathbb{Z}$$

cannot be negative