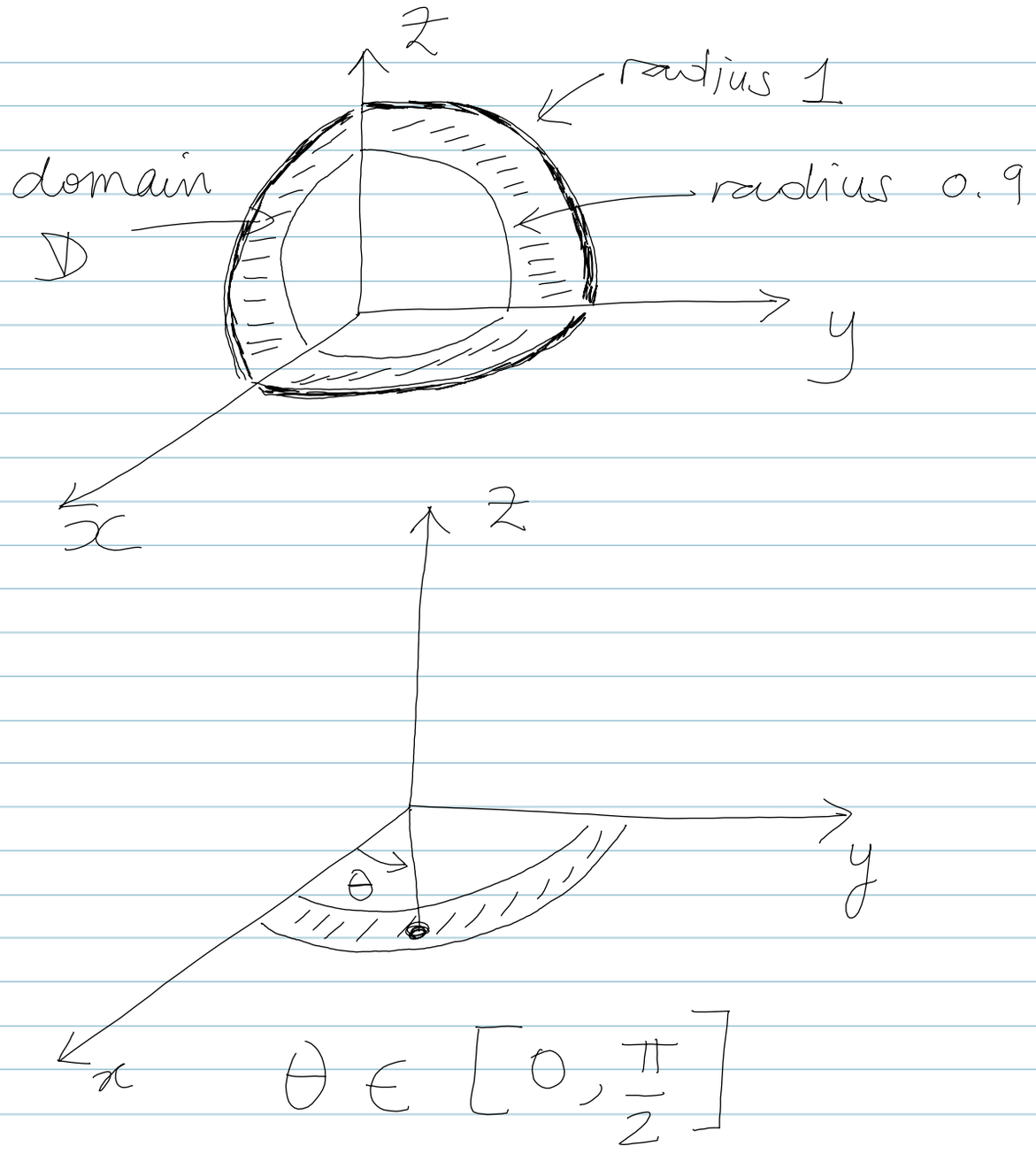
Except from an exam

We consider two spheres with center $(0,0,0)$ and radii 0.9 and 1 respectively. Let D be the set of points (x,y,z) located between the two spheres, with additional condition that $x \geq 0$, $y \geq 0$, $z \geq 0$.

Compute $\iiint_D \frac{y}{x^2+y^2+z^2}\, dx\,dy\,dz$

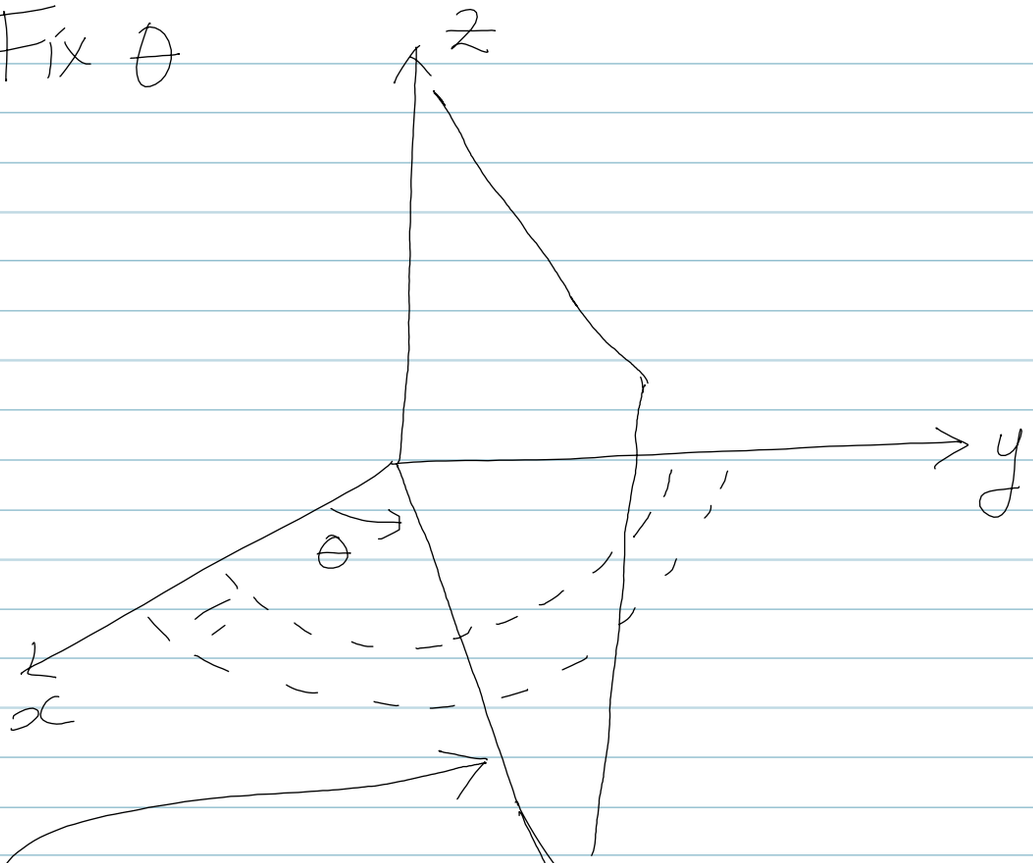
Answer:

We consider a change to spherical coordinates.

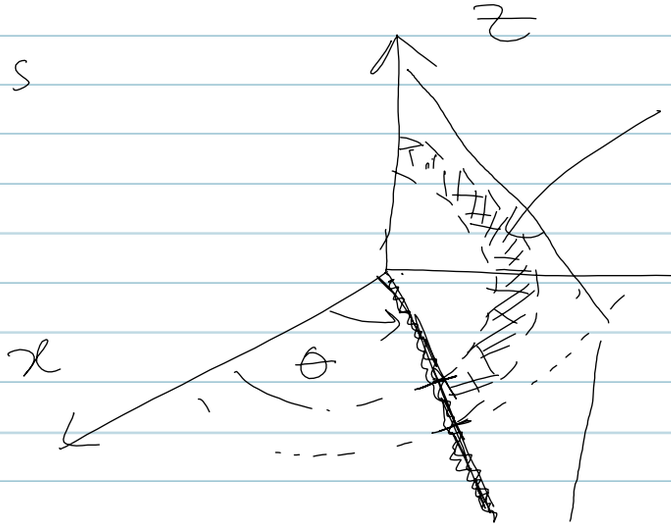


$$\theta \in \left[0, \frac{\pi}{2}\right]$$

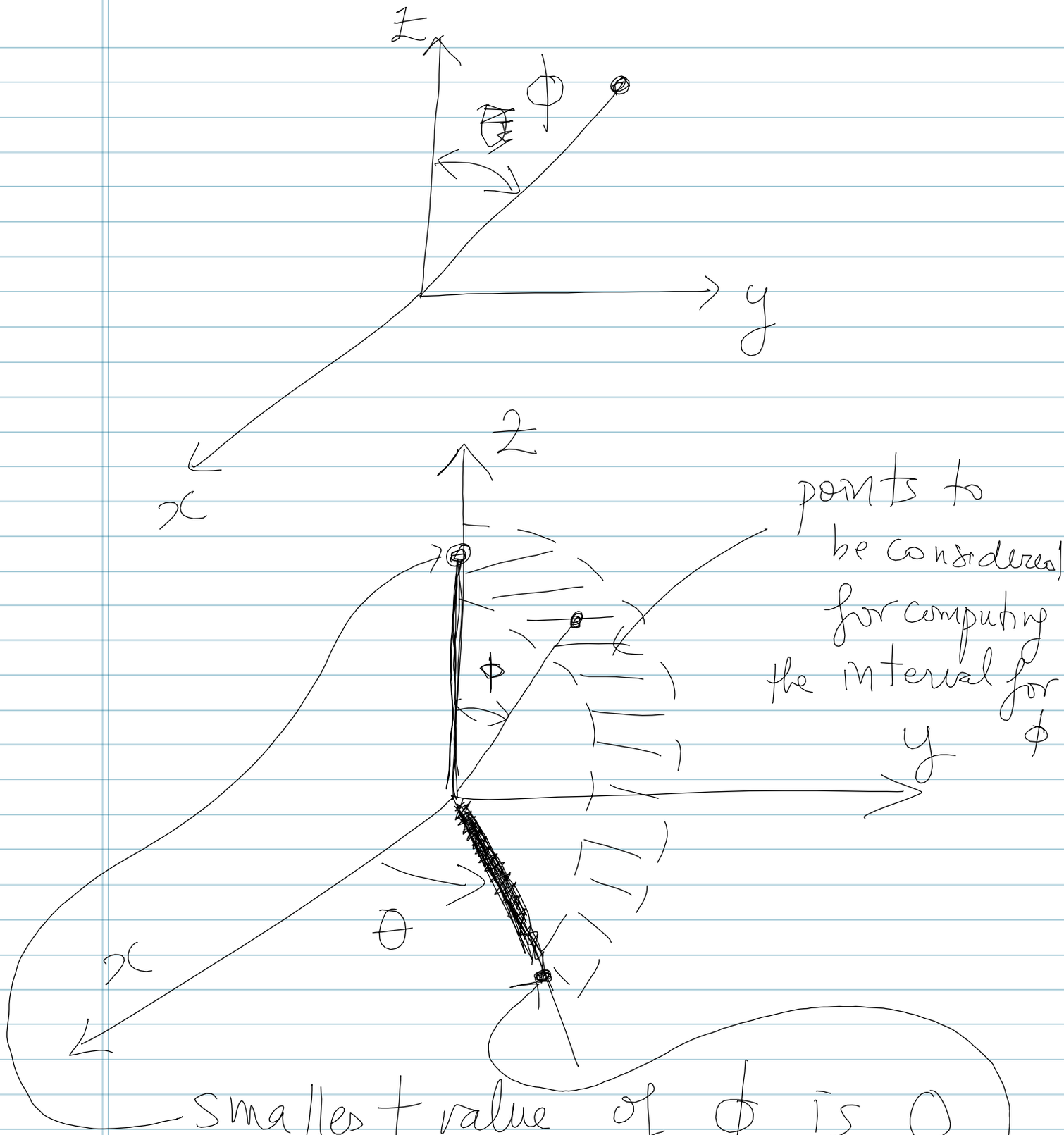
Fix θ



what are the points with the same θ? this is the plane that contains this line and the z-axis

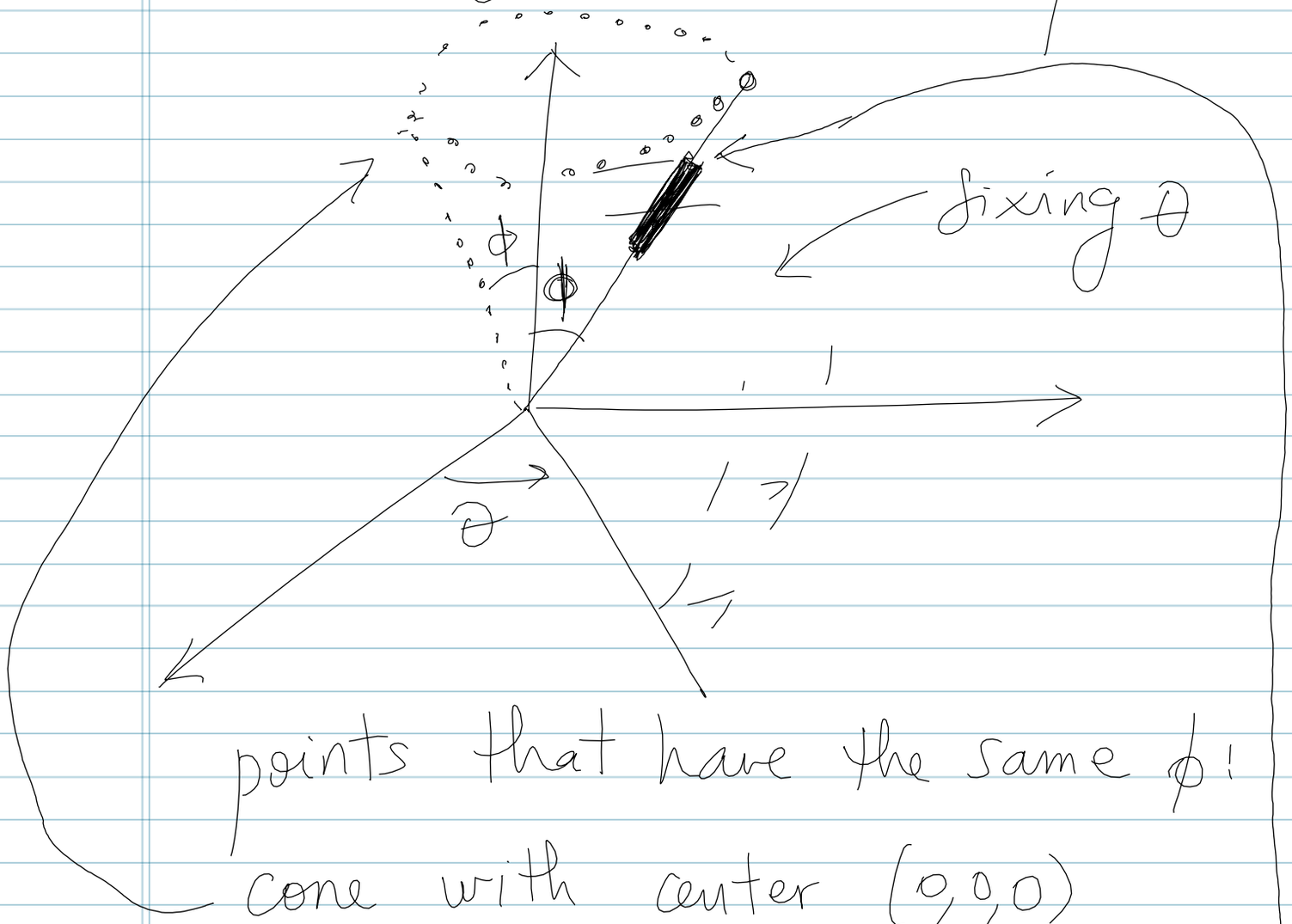


these are the points that you have to consider to find the interval for ϕ



$\phi \in [0, \frac{\pi}{2}]$

Fix θ and Fix ϕ

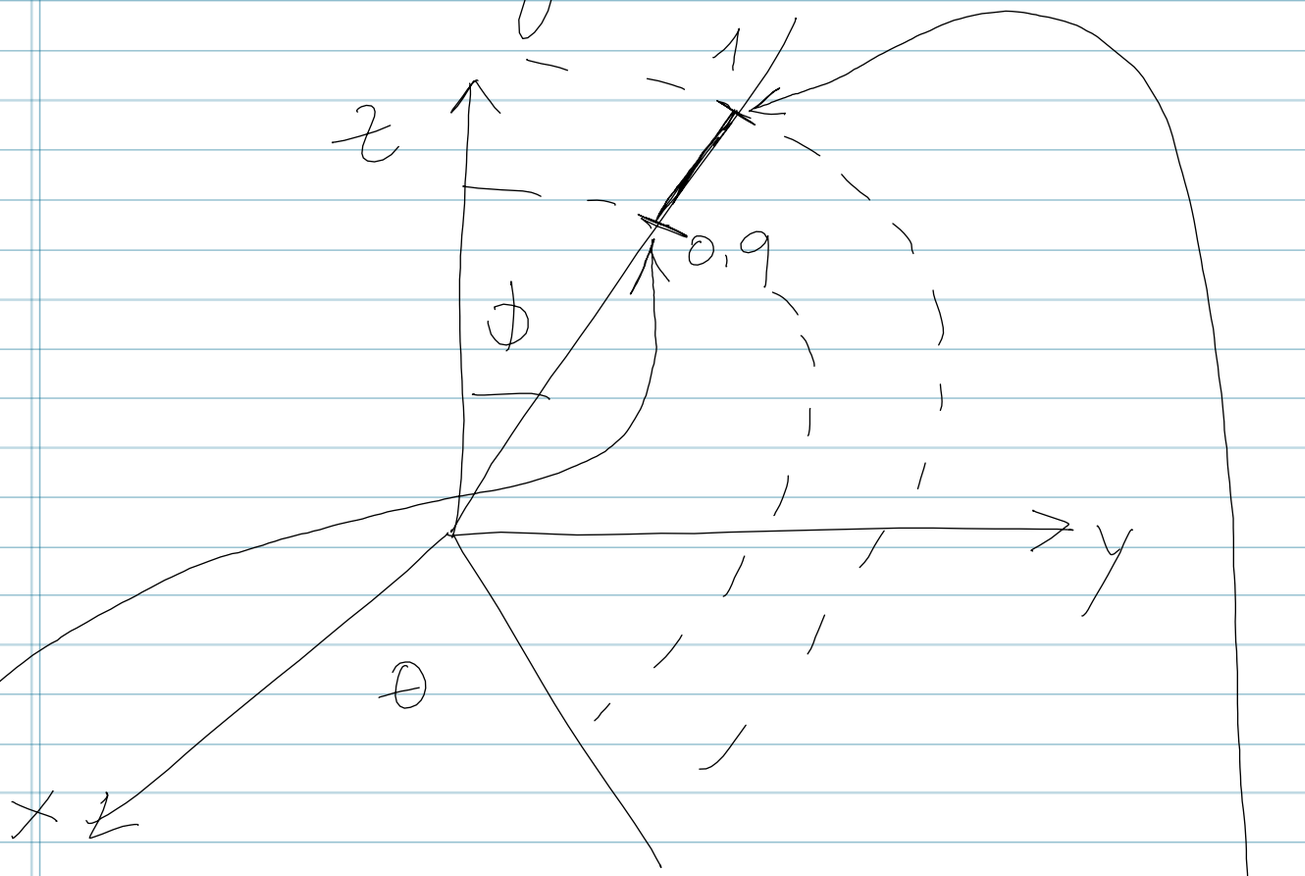


points that have the same ϕ!

cone with center (0,0,0)

$\Rightarrow$ the points that have the same θ and the same ϕ are those that belong to this segment

Consider all the points of this segment



ρ is the distance from the origin of the coordinates to your point.

Smallest $\rho = 0.9$

Largest $\rho = 1$

$$\rho \in [0.9, 1]$$

$$\rho = \sqrt{x^2+y^2+z^2}$$

$$y = \rho \sin(\phi)\sin(\theta)$$

so that

$$\iiint_D \frac{y}{\sqrt{x^2+y^2+z^2}}\, dx\, dy\, dz$$

$$= \iiint_{D'} \frac{\rho \sin(\phi)\sin(\theta)}{\rho^2} \cdot \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$D' = \underbrace{\left[0, \frac{\pi}{2}\right]}_{\theta} \times \underbrace{\left[0, \frac{\pi}{2}\right]}_{\phi} \times \underbrace{[0.9, 1]}_{\rho}$$

$$(1) = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_{0,9}^{1} \rho \sin^2(\phi) \sin(\theta) \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin^2(\phi) \sin(\theta) \left[\frac{\rho^2}{2}\right]_{0,9}^{1} d\phi \, d\theta$$

Observe that $\left[\frac{\rho^2}{2}\right]_{0,9}^{1} = 0,095$

and $\int \sin^2(\phi) \, d\phi = \frac{\phi}{2} - \frac{\sin(2\phi)}{4}$

Thus

$$(1) = 0,095 \int_0^{\frac{\pi}{2}} \sin(\theta) \left[\frac{\phi}{2} - \frac{1}{4}\sin(2\phi)\right]_0^{\frac{\pi}{2}} d\theta$$

$$= 0,095 \cdot \frac{\pi}{4} \int_0^{\pi/2} \sin(\theta) \, d\theta$$

$$= 0,095 \cdot \frac{\pi}{4} \left[-\cos(\theta) \right]_0^{\pi/2}$$

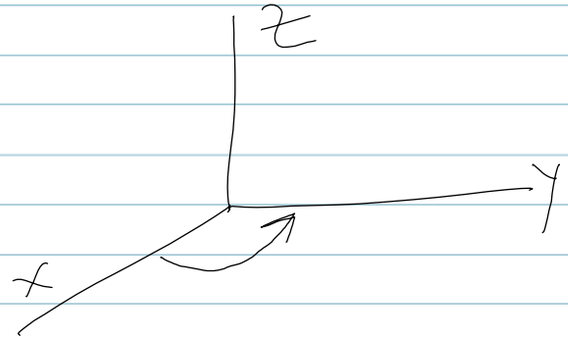
$$\simeq 0,0746 .$$

Student's answer

$$\int \sin^2(\phi)\, d\phi = \frac{\sin^3(\phi)}{3}$$

↑ incorrect

$$\theta \in [0, 2\pi]$$



$x \geq 0$

$y \geq 0$ $\quad z \geq 0$