Excerpt from an exam

Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be known C^1 functions such that:

$\forall x \in \mathbb{R}$, $f(x) \neq 0$ and $\forall x \in \mathbb{R}$, $g(x) > 0$.

Define the functions $h_1 : \mathbb{R}^2 \to \mathbb{R}$ and $h_2 : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$\forall (x,y) \in \mathbb{R}^2$, $h_1(x,y) = f\left(f(-x) + g^2(y)\right)$

$\forall (x,y) \in \mathbb{R}^2$, $h_2(x,y) = \left(g(-x)\right)^{\frac{1}{f(y)}}$

We consider the vector field $h: \mathbb{R}^2 \to \mathbb{R}^2$ defined as: $\forall (x,y) \in \mathbb{R}^2$ we have

$h(x,y) = \left(h_1(x,y), h_2(x,y)\right)$

(a) Find the domain of h

(b) Find the domain on which we can compute curl(h)

(c) Compute curl(h)

Answer:

(a) $h_2(x,y) = e^{\frac{1}{f(y)} \cdot \ln(g(-x))}$

We know that $g(-x) > 0 \quad \forall x \in \mathbb{R}$.

Also $f(y) \neq 0 \quad \forall y \in \mathbb{R}$

This means that h_2 is well defined for all $(x,y) \in \mathbb{R}^2$.

h_1 is well defined for all $(x,y) \in \mathbb{R}^2$.

Thus, h is well defined for all $(x,y) \in \mathbb{R}^2$.

$$\boxed{\operatorname{dom}(h) = \mathbb{R}^2}$$

(b) Since the functions f and g are C^1, it follows that h_1 and h_2 are C^1 as combinations of C^1 functions.

Thus h is also C^1, so all partial derivatives of h exist for all $(x,y) \in \mathbb{R}^2$.

$$\operatorname{dom}(\operatorname{curl}(h)) = \mathbb{R}^2$$

(c) $\operatorname{curl}(h) = \left(\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y}\right) \cdot e_3$

where $e_3 = (0, 0, 1)$

$$\frac{\partial h_1}{\partial y}(x,y) = \frac{\partial\left[f(-x) + g^2(y)\right]}{\partial y} \cdot f'\left(f(-x) + g^2(y)\right)$$

$$= 2g'(y)\, g(y) \cdot f'\left(f(-x) + g^2(y)\right)$$

$$\frac{\partial h_2}{\partial x}(x,y) = \frac{\partial\left[\frac{1}{f(y)} \cdot \ln(g(-x))\right]}{\partial x} \cdot e^{\frac{1}{f(y)} \cdot \ln(g(-x))}$$

$$= \frac{1}{f(y)} \cdot \frac{\partial\left[\ln(g(-x))\right]}{\partial x} \cdot e^{\frac{1}{f(y)} \cdot \ln(g(-x))}$$

$$= -\frac{1}{f(y)} \cdot \frac{g'(-x)}{g(-x)} \cdot \left(g(-x)\right)^{\frac{1}{f(y)}}$$

$$= -\frac{g'(-x)}{f(y)} \cdot \left(g(-x)\right)^{\frac{1}{f(y)}-1}$$

student's answer

$$h_2(x,y) = \left(g(-x)\right)^{\frac{1}{f(y)}} \quad \} \rightarrow$$

$$\mathrm{Dom}(h_2) = \mathbb{R} \times \mathbb{R} \setminus \{f(y)=0\}$$

By assumption we have $f(x) \neq 0 \ \forall x \in \mathbb{R}$

$f(y)$ cannot be zero

$\{f(y)=0\}$ this is a correct way to write a set.

Correct way:

$\{(x,y)\in\mathbb{R}^2 / f(y)=0\}$
$=\emptyset$