Control & Guidance

2011

Enginyeria Tècnica d'Aeronàutica
esp. en Aeronavegació

Escola d'Enginyeria de Telecomunicació
i Aeroespacial de Castelldefels

Adeline de Villardi de Montlaur
Marc Diaz Aguiló

Digital Control

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH

Escola Politècnica Superior de Castelldefels
UNIVERSITAT POLITÈCNICA DE CATALUNYA

Digital control

1. Introduction to discrete systems
2. Z transform
3. Z transfer function
4. Digital control tools
5. Design method with dead beat response

1. Introduction to discrete systems

a. Revision of the sampling theorem

Digital signals present great advantages when transmitted and/or processed:

- higher immunity to noise
- easier to process
- multiplexing easiness (multiple digital data streams are combined into one signal over a shared medium),…
- obvious tendence to the usage of digital controllers (microcontrollers, PIC, or even Computers)

a. Revision of the sampling theorem

As a consequence:

There is an latent interest in changing analog signals to digital signals

Analog systems:
-- continuous time
-- continuous amplitude

Discrete system:
-- discrete time
-- quantized amplitude

Tasks to be performed:
Discretize in time: “sampling”.
Discretize in amplitude: “quantization”.

a. Revision of the sampling theorem

In order to analyze the properties of the discretized signal, we use the Fourier transform, this transformation is defined on an infinite continuous interval.

Therefore, in order to operate on discrete signals, we lead to a digital data processing problem.

The signal will be replaced by samples taken at a determined rate.

The objective is to represent the continuous signal and process it without any information loss.

UNIVERSITAT POLITÈCNICA DE CATALUNYA BARCELONATECH
Escola Politècnica Superior de Castelldefels
UNIVERSITAT POLITÈCNICA DE CATALUNYA

a. Revision of the sampling theorem

The digital sampling of an analogical signal needs a discretization both in the temporal domain (*time sampling*) and in the amplitude one (*quantization*).

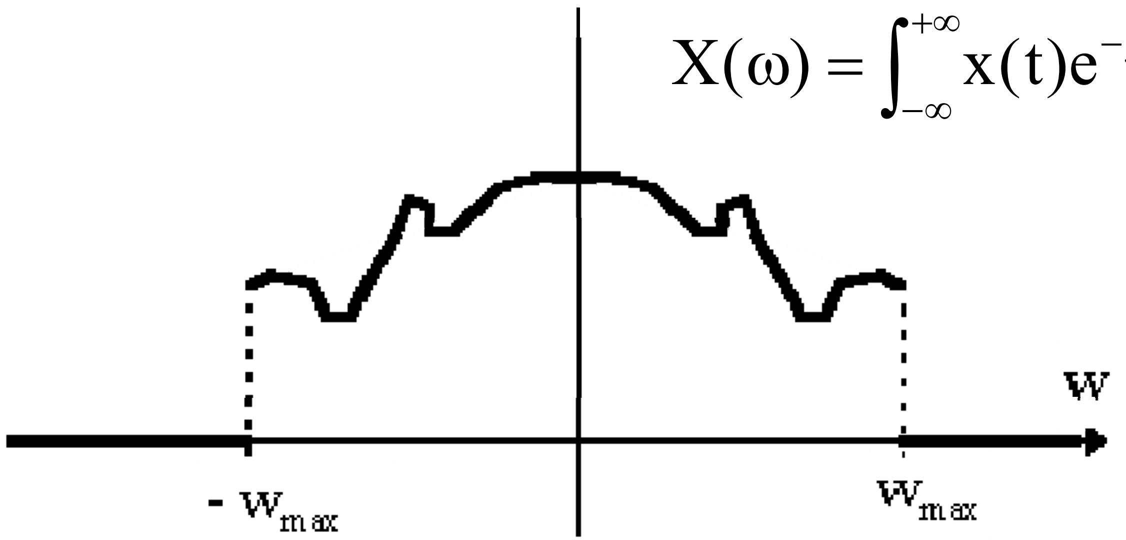
There are different ways to mathematically describe the temporal discretization process of a signal that is continuous in time. We will analyze some of them during these sessions.

a. Revision of the sampling theorem

Theoretical sampling (ideal)

Given *x(t)* a real and continuous signal with limited band, which spectrum *X(ω)* is null for |*ω*| > *Wmax*

$$X(\omega)=\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt$$



UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Escola Politècnica Superior de Castelldefels
UNIVERSITAT POLITÈCNICA DE CATALUNYA

1. Introduction to discrete systems

a. Revision of the sampling theorem

Theoretical sampling (ideal)

and we consider the ideal sampling wave with a T_s period:

$$s_d(t) = \sum_{m=-\infty}^{+\infty} \delta(t - mT_s)$$

the product $x(t) \cdot s_d(t)$ is a wave formed by Dirac deltas whose amplitude is the same as the x(t) samples:

$$x_d(t) = x(t) \times s_d(t) = x(t) \times \sum_{m=-\infty}^{+\infty} \delta(t - mT_s) = \sum_{m=-\infty}^{+\infty} x(mT_s) \times \delta(t - mT_s)$$

Drawing

a. Revision of the sampling theorem

Theoretical sampling (ideal)

Consequently, its spectrum in the time domain is:

$$X_d(\omega) = f_s \sum_m X(\omega - m2\pi f_s) \quad with \ f_s = \frac{1}{T_s}$$

$$X_d(f) = f_s \sum_m X(f - mf_s) \quad with \ f_s = \frac{1}{T_s}$$

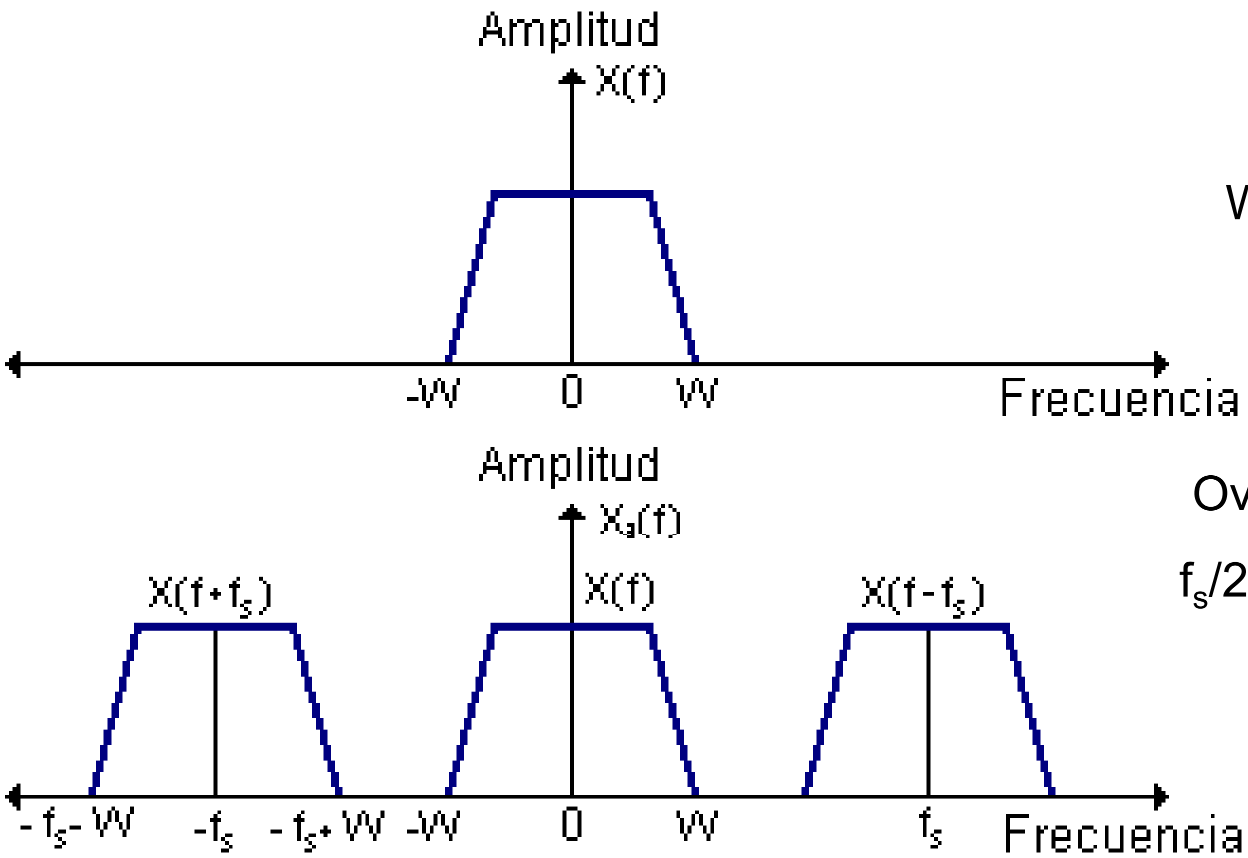
The spectrum of the signal $x_d(t)$ is the replica of the spectrum of the signal x(t) at each multiples of f_s.

Note the scale factor f_s.

1. Introduction to discrete systems

a. Revision of the sampling theorem

Theoretical sampling (ideal)



$$f = \frac{\omega}{2\pi}$$

W = f_{max} of the spectrum

Overlap condition

$f_s/2 > W \rightarrow f_s > 2W$

a. Revision of the sampling theorem

Theoretical sampling (ideal)

Sampling theorem (Nyquist): every signal with finite energy and limited bandwidth can be expressed in a unique mode depending on its samples or instant values taken in regular intervals T_s , with T_s as:

$$f_s = \frac{1}{T_s} \geq 2W$$

being W the signal maximum frequency

Sinusoidal drawing

a. Revision of the sampling theorem

Theoretical sampling (ideal)

The minimum sampling frequency: $f_s = 2\ W$ (Hz) is called:

→ **Nyquist frequency**

In case that the signal is sampled at a lower frequency, the sampled signal spectrum will overlap and the original message will not be able to be recovered.

b. Interpolation formula

Objective: recovery the continuous signal.

By a correct interpolation process, we can mathematically define a continuous-time signal $x(t)$ from the discrete samples $x[n]$

Original message can be recovered using an ideal low-pass filter which cut-off frequency will be W:

$$\Pi_{x_d}(f) = \begin{cases} 1 \text{ if } |f| \leq W \\ 0 \text{ else} \end{cases}$$

x(t) is recovered by the inverse Fourier transform:

$$x(t) = \int_{-\infty}^{+\infty} X_d(f) e^{j2\pi ft} df$$

Then, because of the bandwidth limitation:

$$x(t) = \int_{-W}^{+W} X_d(f) e^{j2\pi ft} df$$

b. Interpolation formula

Computing: in the freq domain:

$$X(f) = X_d(f) \cdot \Pi_{x_d}(f)$$

sinc function: $\sin(\pi x)/(\pi x)$

$$\Pi_{x_d}(f) \rightarrow \mathrm{sinc}(t/T_s)$$

$$x(t) = x_d(t) \ * \ \mathrm{sinc}(t/T_s)$$

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \delta(t - nT_s) * \ \mathrm{sinc}(t/T_s) =$$

$$= \sum_{n=-\infty}^{\infty} x[n] \cdot \ \mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

b. Interpolation formula

Finally,

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

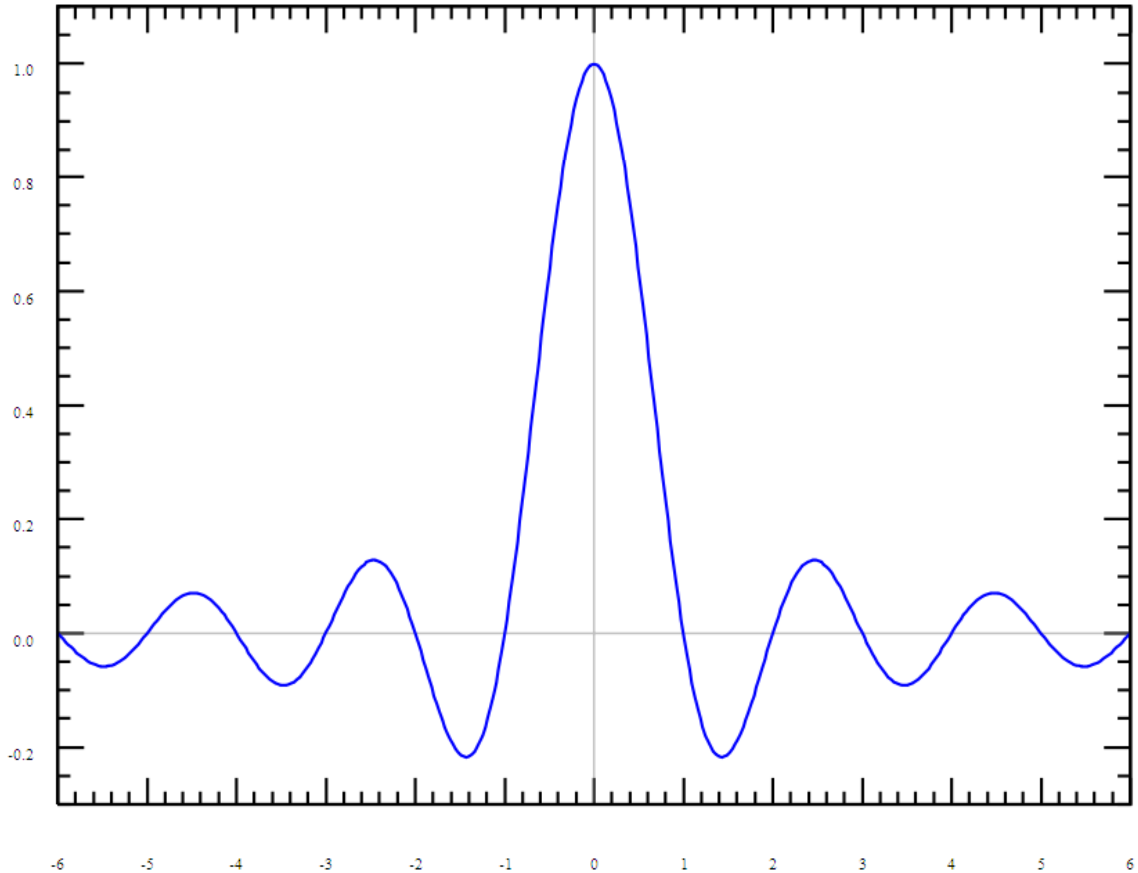
→ Each sample value multiplied by sinc function scaled so that zero-crossings of sinc occur at sampling instants and that sinc function's central point is shifted to the time of that sample, nT.

→ All of these shifted and scaled functions added together to recover the original signal.

→ The scaled and time-shifted sinc functions are continuous making the sum of these also continuous, so the result of this operation is a continuous signal.

1. Introduction to discrete systems

b. Interpolation formula



normalized sinc function: $\sin(\pi x) / (\pi x)$ ... showing the central peak at $x=0$, and zero-crossings at the other integer values of x.

b. Interpolation formula

Problems:

• all the samples are needed in order to re-obtain $x(t)$: in practice only a finite number of samples will be considered and are available → truncation error.

• generally, real signals spectrum tends to 0 for $f > Wmax$ but they are not exactly zero.

• ideal sampling → physically unfeasible (use a sampling wave)

• non-ideal filter is used (filters with inifinite derivative are not feasible).

b. Interpolation formula

Ideal reconstruction process not possible to be implemented.

(since it implies that each sample contributes to the reconstructed signal at almost all time points, requiring summing an infinite number of terms)

→ approximation of the sinc functions, finite in length (that means that they cannot be finite in frequency). This leads to the *interpolation error*.

→ practical digital-to-analog converters produce neither scaled delayed sinc functions, but a sequence of scaled and delayed rectangular pulses: *zero-order hold filter.*

c. Practical sampling

The ***real*** sampling systems differ from the theoretical ones in:

- the sampling wave is composed by a pulse series where each pulse has a non-zero duration.

- the affected signals are not strictly bandwidth limited and they can not be, because they are time-limited signals (not infinite).

c. Practical sampling

Indeed, the sampling wave generally responds to a function like:

$$s(t) = \sum_{m} p(t - mT_s)$$

composed by rectangular pulses with amplitude p and the condition:

$$p(t) = 0, \quad \forall \, |t| \geq \frac{\tau}{2}, \quad \text{with } \tau \leq T_s$$

in order to avoid overlap between basic pulses.

Drawing

c. Practical sampling

There exists mainly 2 types of practical sampling:

The instantaneous sampling, Sample & Hold in which the following signal is formed:

$$x_{p_i}(t) = \sum_m x(mT_s)p(t - mT_s)$$

which result is a pulse series, where every pulse has a constant amplitude taken as the instantaneous value of $x(mT_s)$.

The natural sampling, that is like: $x_{p_n}(t) = x(t)\sum_m p(t - mT_s)$

in which each pulse varies with $x(t)$ in the existence interval

→ In both cases the sampling theorem remains valid even if the ideal sampling wave is not used → $f_s > 2W$

c. Practical sampling

In the **instantaneous sampling** case we can write:

$$x_{pi}(t) = \sum_m x(mT_s)\, p(t - mT_s) = p(t) \sum_m x(mT_s)\, \delta(t - mT_s)$$

Since the inner expression of the sum is the one obtained for the ideal sampling case, it can be written:

$$X_{pi}(f) = f_s\, P(f) \sum_m X(f - mf_s)$$

spectrum affected by *P(f)* value

the effect can be reduced shortening the duration of the sampling pulse

Advantages:
- easy to do with "Sample & Hold" circuits
- immune to noise
- pulse form is not important

1. Introduction to discrete systems

c. Practical sampling

In the **natural sampling** case the transform is:

$s(t) = \sum_m p(t - mT_s)$ como $p(t - mT_s) = p(t) * \delta(t - mT_s)$ se tiene que

$$s(t) = p(t) \sum_m \delta(t - mT_s)$$

whose transform is:

$$S(f) = P(f)\, f_s \sum_n \delta(f - nf_s) = f_s \sum_n P(nf_s)\delta(f - nf_s)$$

$$X_{pn}(f) = X(f) * S(f) = X(f) * \left[f_s \sum_n P(nf_s)\delta(f - nf_s) \right] =$$

$$= f_s \sum_n P(nf_s) X(f - nf_s)$$

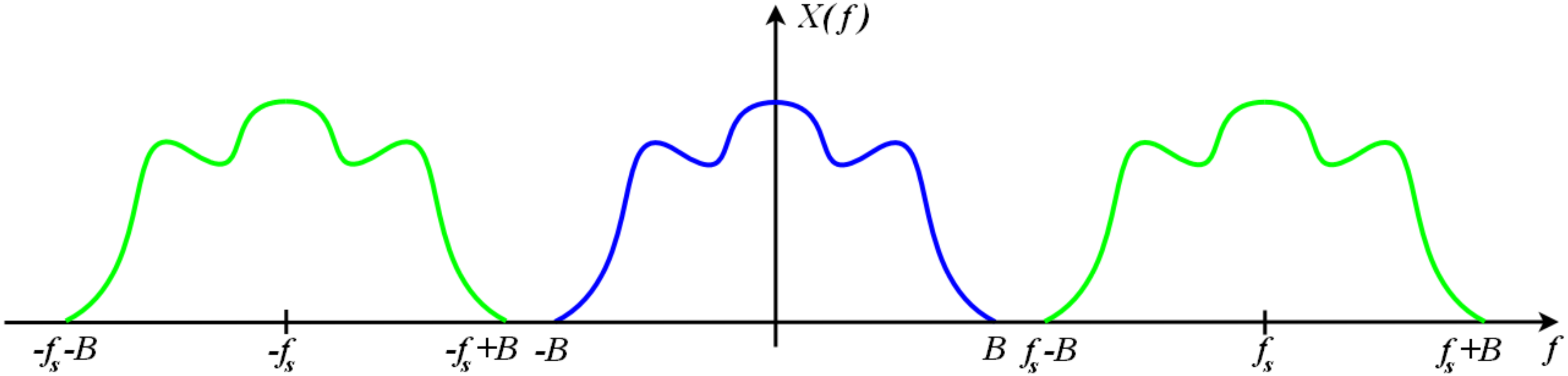
Identical result as the one obtained with an ideal sampling wave formed by a Dirac delta series, but affected by a constant coefficient or scale factor $P(n\, f_s)$.

Drawing

1. Introduction to discrete systems

d. Aliasing

Hypothetical spectrum of a properly sampled bandlimited signal (blue) and images (green) that do not overlap. A low-pass filter can remove the images and leave the original spectrum, thus recovering the original signal from the samples

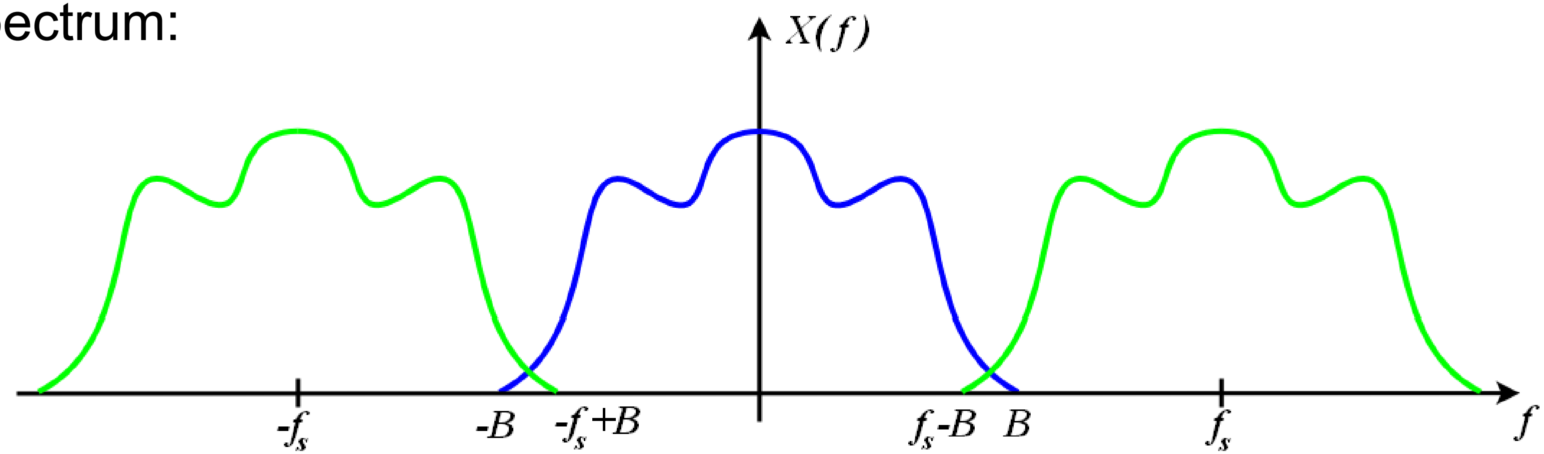


For practical purposes, there can not be a strict limitation of the analyzed bandwidth, because real signals have a finite length

d. Aliasing

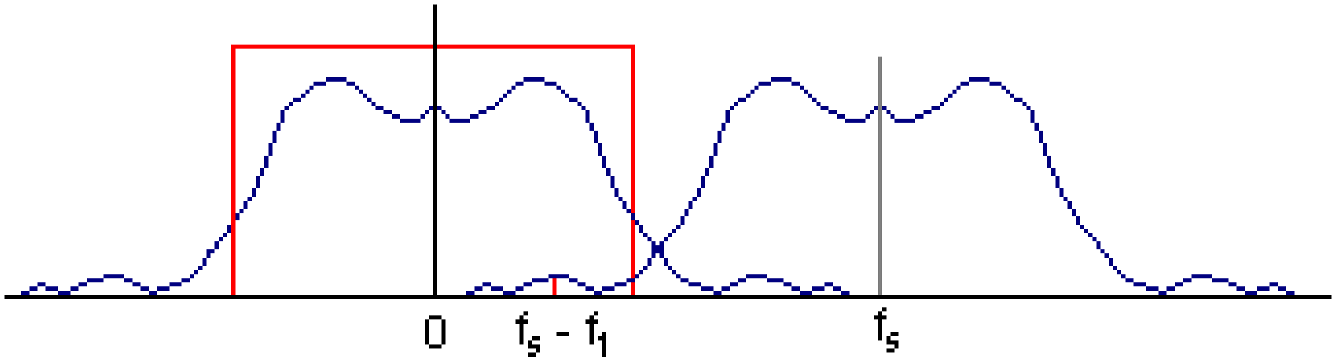
Hypothetical spectrum of insufficiently sampled bandlimited signal (blue), $X(f)$, where the images (green) overlap. This type of spectrum can be considered as bandwidth limited one if the content that exceeds the interval $(-B, B)$ is small, or barely significant.

When this type of signal is sampled, an overlap is inevitably created in the spectrum:



d. Aliasing

In the process of the signal reconstruction, the frequencies of the spectrum centered in f_s, lower than $f_s – B$, that were originally out of the B bandwidth limited now appear inside.



This phenomenon is named aliasing.

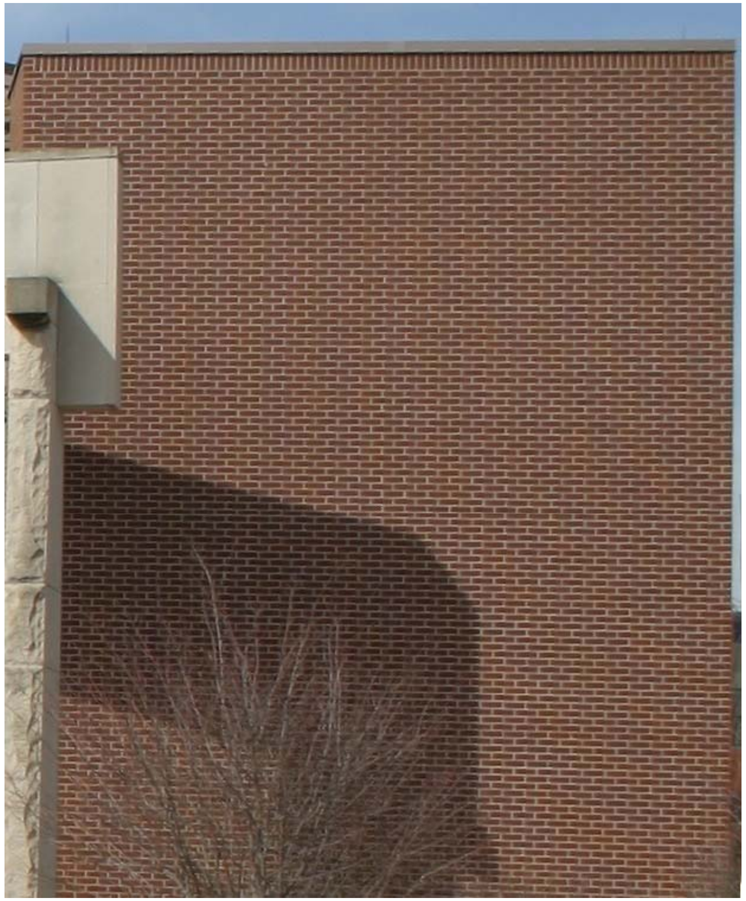
The only way to avoid this effect is to properly increase the sampling frequency so that the components out of the taken bandwidth become very small and their influence is hardly perceptible.

(in practice about 5 to 10 times f_1)

1. Introduction to discrete systems

d. Aliasing

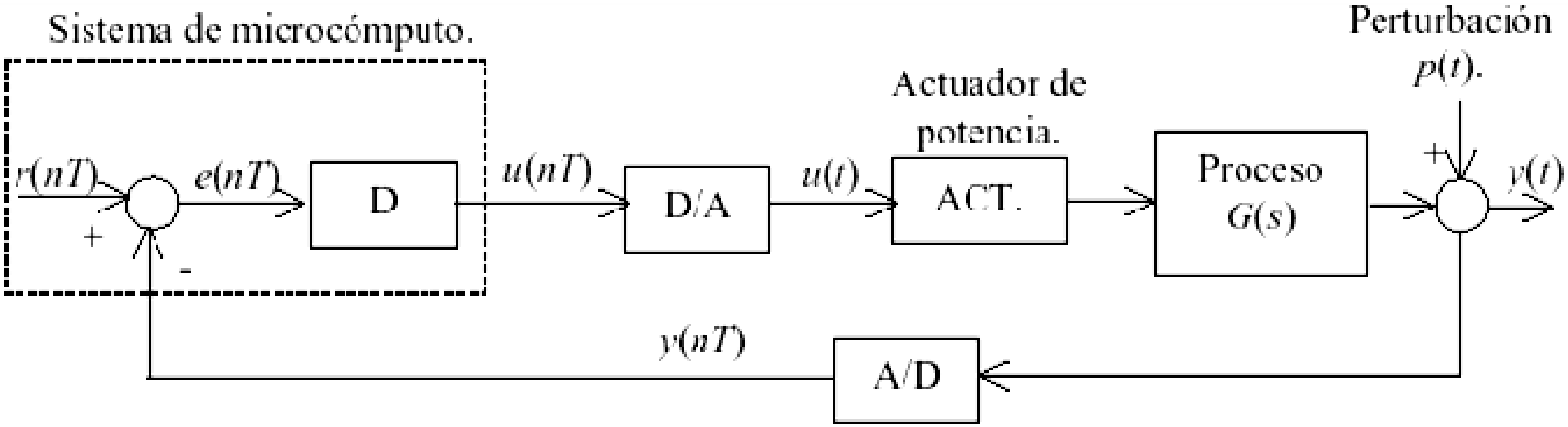
Subsampled image of brick wall



Properly sampled image of brick wall

1. Introduction to discrete systems

e. Digital control diagram



• $y(t)$, the signal to be controlled, is sampled through an A/D, analogical-digital converter, and compared with the reference value $r(nT)$ stored in a memory position of the micro computation system (where the digital controller is implemented)

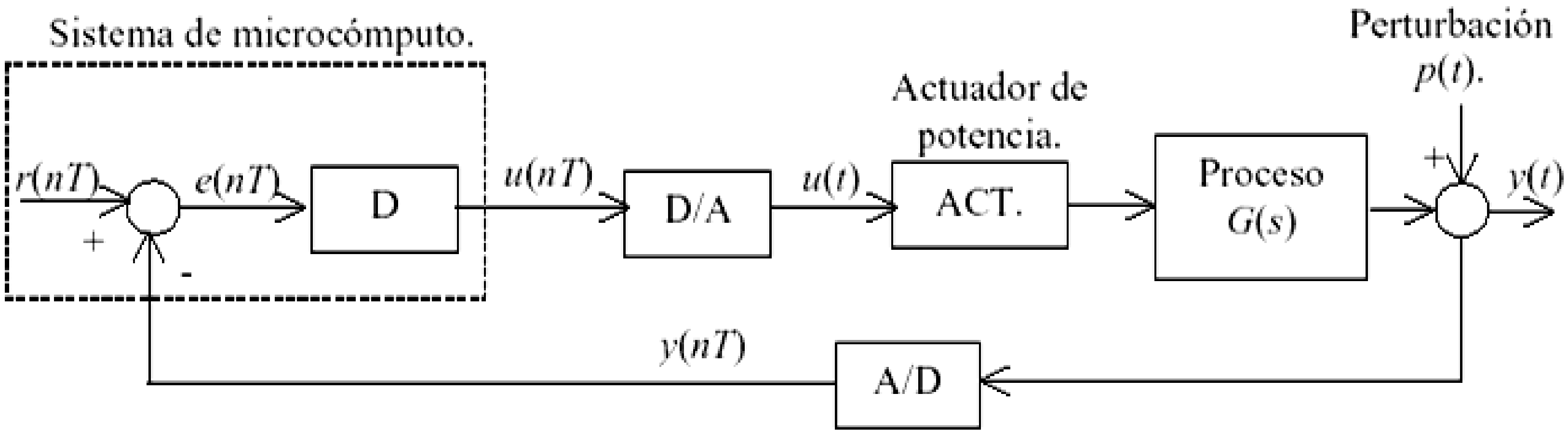
• the result of this comparison is the discrete error signal. This is processed by the microcomputer in order to generate a discrete control signal $u(nT)$ which is transformed into an analogical one through a D/A converter.

• operation sequence made every T_s seconds, being T_s the sampling period

e. Digital control diagram

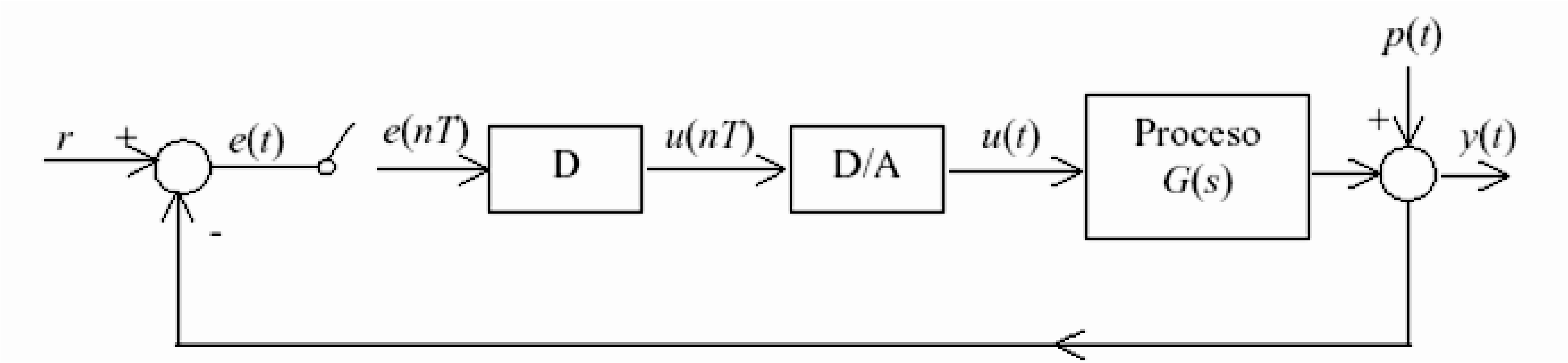
Two signal types can be distinguished:

- **continuous or analog signals**: defined for every time instant ($u(t)$, $y(t)$, $p(t)$).
- **discrete time signals**: only defined in the time instants $t = nT_s$, being n an integer number and T_s the sampling period ($r(nT)$, $e(nT)$, $u(nT)$).



e. Digital control diagram

From the point of view of analysis and design, the following diagram is equivalent:



e. Digital control diagram

In order to analyze the behavior of this system using the mathematical tools that are normally used in analogical systems, we notice that:

The Laplace transform is not defined for a signal that is only defined at specific time samples

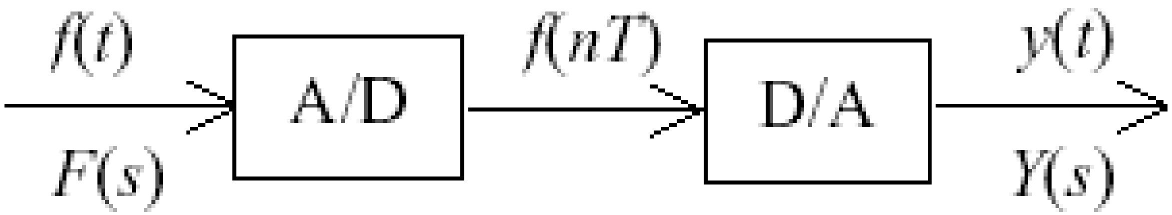
Thus, not all the blocks can be modeled using Laplace transfer functions

→ we need to model the following part of the digital control diagram:

A/D converter – digital controller - D/A converter

e. Digital control diagram

A/D – D/A part to be modeled:

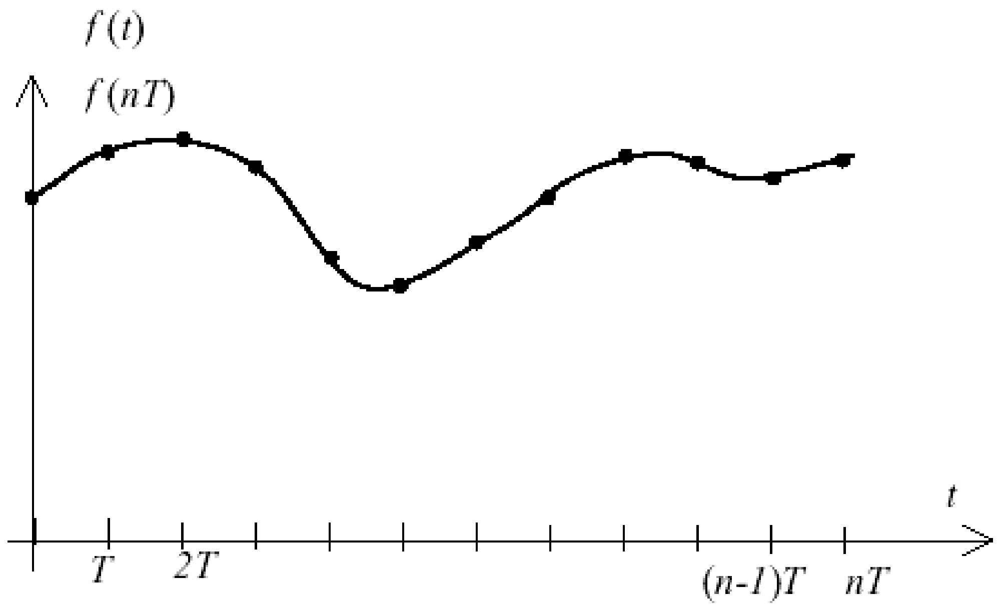


e. Digital control diagram

A/D converter:

generates an impulse series, each of them being weighted by the analogical signal value at the corresponding time $t=nT_s$

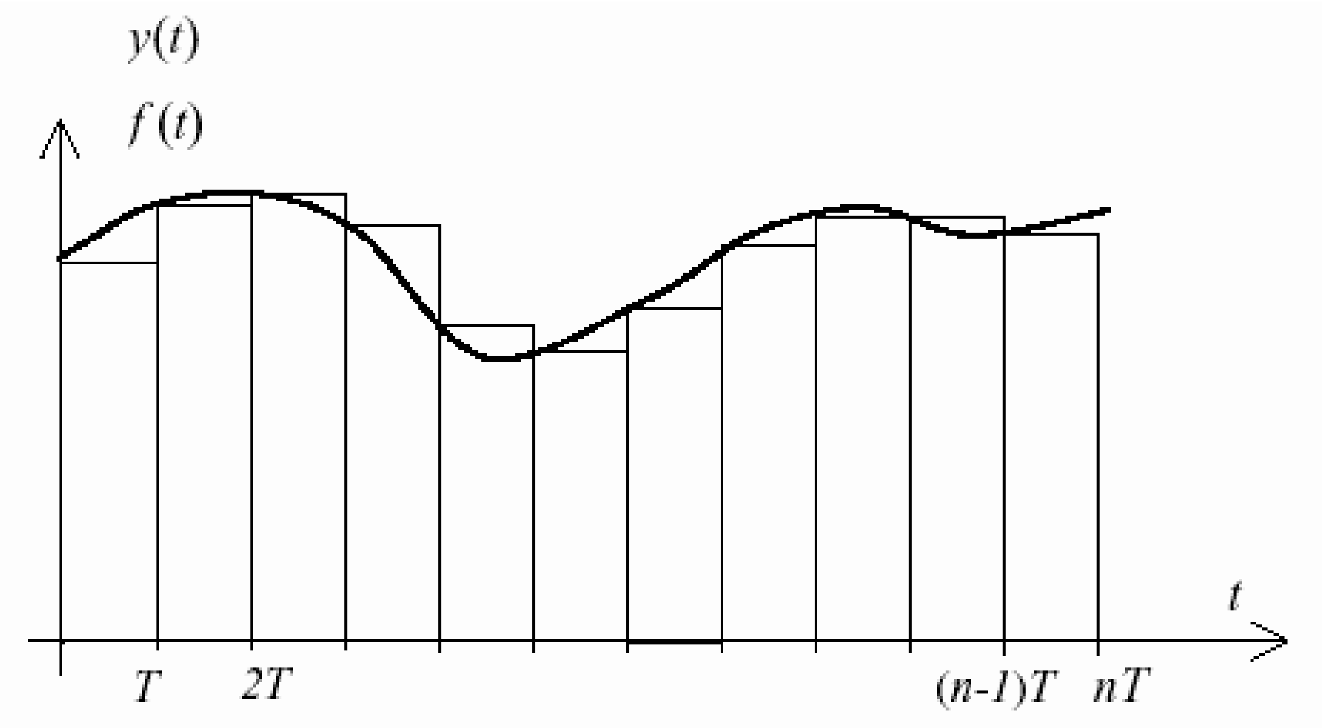
→ **sampler**



e. Digital control diagram

D/A converter:

signal re-constructor converts the impulse series into a stepped signal



e. Digital control diagram

Digital Controller:

It processes the entry signal providing every T_s seconds and it generates a corrected impulse to act on the system.

→ need of a tool to process discrete signals: **z-transform**

f. Digital and analog controllers

- Digital Controllers only operate on numbers, they can handle non-linear control equations that involve complicated calculations or logical operations.
- Larger variety of control laws can be used with Digital Controllers.
- Digital Controllers can execute complex calculations at constant accuracy and at high speed.
- Due to the availability of cheap μ-computers, Digital Controllers are used in the vast majority of control systems.

1. Introduction to discrete systems

f. Digital and analog controllers

- Analog controllers must represent the variables in an equation using continuous physical amounts.
- Analog controllers must be built with physical components such as transistors, capacitors, inductors, resistances...
- The cost of the Analog Controller increases quickly as the calculation complexity increases, if a constant accuracy has to be maintained.

f. Digital and analog controllers

Additional advantages of the Digital Controllers:

• Digital components (A/D converters, D/A, etc..) are robust, highly trustworthy and usually compact and light.

• Digital systems are scalable.

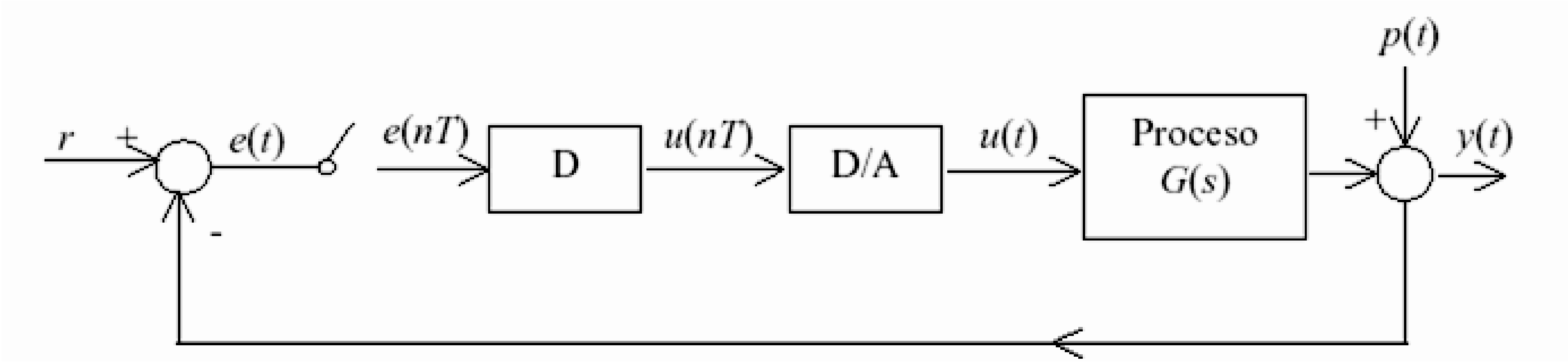
• High sensitivity and cheaper.

• Less sensitive to noise signals.

• Flexible, allow programming changes.

• Less prone to environmental conditions.

2- Z-transform



2. Z-transform

We consider a sampled signal (ideal):

$$f^*(t) = f(0)\delta(t) + f(1)\delta(t-1T) + f(2)\delta(t-2T) + \ldots + f(k)\delta(t-kT)$$

Using:

- the Laplace transform $\delta(t) \rightarrow 1$
- the temporal delay property of the Laplace transform:

$$f(t-nT) \xrightarrow{L} e^{-nT\cdot s} \times F(s)$$

$$F^*(s) = L\left[f^*(t)\right] = f(0) + f(1)e^{-Ts} + \ldots + f(k)e^{-kTs}$$

$$= \sum_{k=0}^{+\infty} f(k)e^{-kTs}$$

2. Z-transform

Given the change of variable : $z = e^{Ts}$

The z transform is obtained for a time function x(t) or x(kT) (T sampling period) :

$$F(z) = Z[f(t)] = f(0) + f(T)z^{-1} + \ldots + f(kT)z^{-k}$$
$$= \sum_{k=0}^{+\infty} f(kT)z^{-k}$$

and for a number sequence:

$$F(z) = Z[f(k)] = f(0) + f(1)z^{-1} + \ldots + f(k)z^{-k}$$
$$= \sum_{k=0}^{+\infty} f(k)z^{-k}$$

Z transform examples:

Z-transforms of time functions:

- step
- ramp

Sequence Z-transforms:

- 0, 0, 1, 1, 1…
- 0, 2, 5, 1 and then 0
- a^k

Exercices

Usual transforms:

using:

$$F(z) = \sum_{k=0}^{+\infty} f(kT) z^{-k}$$

$f(t)$	$F(s)$	$F(z)$
$u(t)$	$\frac{1}{s}$	$\frac{z}{z-1} = \frac{1}{1-z^{-1}}$
e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}} = \frac{1}{1-e^{-aT}z^{-1}}$
t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$\frac{z\sin(aT)}{z^2-2z\cos(aT)+1}$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$\frac{z(z-\cos(aT))}{z^2-2z\cos(aT)+1}$
a^k		$\frac{z}{z-a}$

Properties and theorems

Linearity:

$$Z[\alpha x(t)+\beta y(t)]=\alpha X(z)+\beta Y(z)$$

Multiplication by a^k:

$$Z[a^k x(k)]=\sum_{k=0}^{+\infty} a^k x(k) z^{-k}$$

$$=\sum_{k=0}^{+\infty} x(k)\left(a^{-1}z\right)^{-k}=X(a^{-1}z)$$

Real translation theorem:

$$Z[x(t-nT)] = z^{-n}X(z)$$

it delays the x(t) function of a time nT

$$Z[x(t+nT)] = z^{n}\left[X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k}\right]$$

it advances the function x(t) of a time nT

Complex translation theorem:

$$Z[e^{-at}x(t)] = X(e^{aT}z)$$

Example

2. Z transform

Initial value theorem:

$$x(0) = \lim_{z\to\infty} X(z)$$

Final value theorem:

Hypothesis: all the poles of X(z) are inside the unitary circle, with the only exception of one pole on z=1 (**stability condition**)

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1}\left[\left(1 - z^{-1}\right)X(z)\right]$$

Inverse z-transform:

Equivalent to the inverse Laplace transform.

Be careful: only the discrete time sequence at the sampling times is obtained from the inverse z-transform (not the continuous signal).

Inverse z-transform:

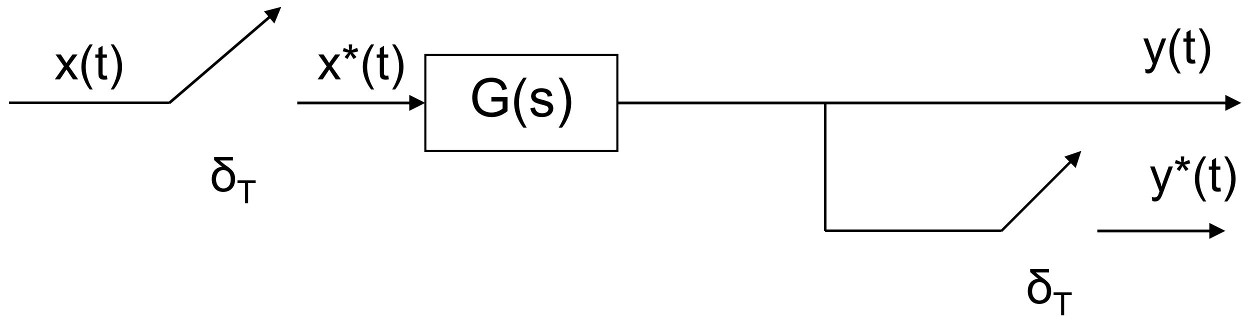
After the decomposition in simple fractions, it is identified in order to get the inverse z-transform (cf. Laplace)

$F(z)$	$F^{-1}(z) = f(kT)$
$\frac{z}{z-1} = \frac{1}{1-z^{-1}}$	$u(kT)$
$\frac{z}{z-e^{-aT}} = \frac{1}{1-e^{-aT}z^{-1}}$	e^{-akT}
$\frac{Tz}{(z-1)^2}$	kT
$\frac{z\sin(aT)}{z^2-2z\cos(aT)+1}$	$\sin(akT)$
$\frac{z(z-\cos(aT))}{z^2-2z\cos(aT)+1}$	$\cos(akT)$
$\frac{z}{z-a}$	a^k

Examples

3. The Z-transfer function

a. Convolution sum



- sampled income signal
- if there is another sampler at the exit, it is synchronized with the entry sampler = both have the same sampling period T
- We need to obtain the relation between x*(t) and y*(t) (i.e. the relation between X(z) and Y(z).

a. Convolution sum

$$x^*(t) = \sum_{k=0}^{+\infty} x(t)\delta(t-kT) = \sum_{k=0}^{+\infty} x(kT)\delta(t-kT)$$

$$y(t) = g(t) * x^*(t) = \sum_{k=0}^{+\infty} x(kT_s)\delta(t-kT_s) * g(t) =$$

$$= \sum_{k=0}^{+\infty} x(kT)g(t-kT)$$

Given g(t): system weight function (response function to δ(t) entry):

$$y(t) = \begin{cases} g(t)x(0) & 0 \le t < T \\ g(t)x(0) \quad + g(t-T)x(T) & T \le t < 2T \\ g(t)x(0) \quad + g(t-T)x(T) \quad + g(t-2T)x(2T) & 2T \le t < 3T \\ \ldots \quad \ldots \quad \ldots \quad \ldots & \ldots \\ g(t)x(0) \quad + g(t-T)x(T) \quad \ldots \quad + g(t-kT)x(kT) & kT \le t < (k+1)T \end{cases}$$

a. Convolution sum

response $y(t)$ to the entry $x^*(t)$ is the sum of the individual impulse responses

Since $g(t)=0$ for $t<0$ is equivalent to $g(t-kT)=0$ for $t<kT$, these equations can be added up:

$$y(t) = g(t)x(0) + g(t-T)x(T) + \dots + g(t-kT)x(kT) \quad \text{for } 0 \le t < (k+1)T$$

$$= \sum_{n=0}^{k} g(t-nT)x(nT) \quad \text{for } 0 \le t < (k+1)T$$

a. Convolution sum

Value of y(t) at the sampling moment t=kT:

$$y(kT) = \begin{cases} \sum_{n=0}^{k} g(kT - nT)x(nT) \\ \sum_{n=0}^{k} x(kT - nT)g(nT) \end{cases}$$

$$\Rightarrow y(kT) = x(kT) * g(kT)$$

b. Z-TF

It links the exit z-transform at the sampling times to the corresponding sampled entry:

$$Z[y(kT)] = \sum_{k=0}^{+\infty} y(kT)z^{-k} = Y(z)$$

$$= \sum_{k=0}^{+\infty}\sum_{n=0}^{+\infty} g(kT - nT)x(nT)z^{-k} \qquad m = k - n$$

$$= \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} g(mT)x(nT)z^{-(m+n)}$$

$$= \sum_{m=0}^{+\infty} g(mT)z^{-m} \sum_{n=0}^{+\infty} x(nT)z^{-n} = G(z)X(z)$$

b. Z-TF

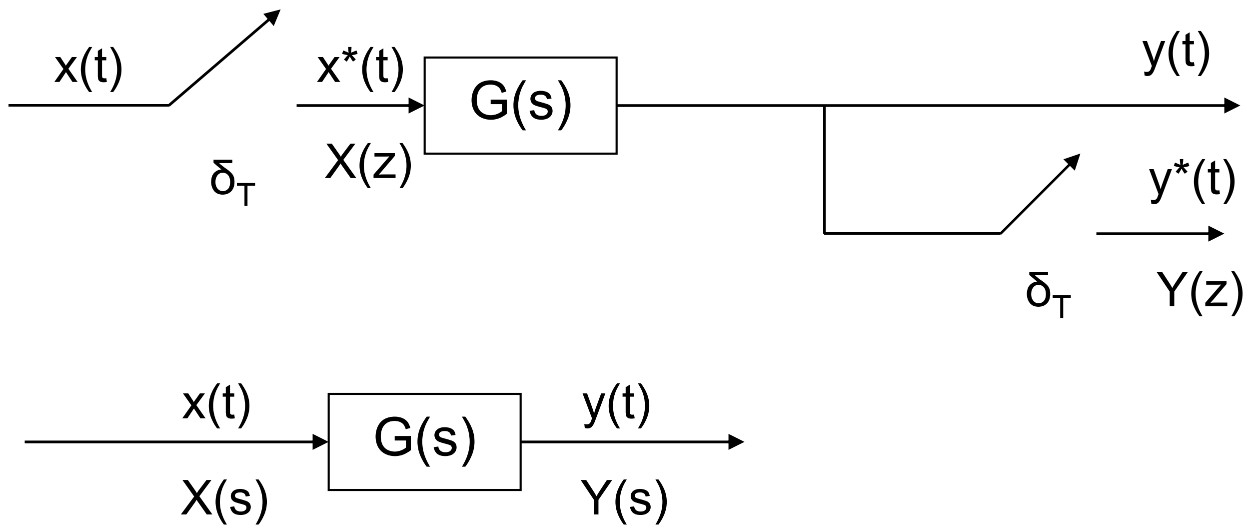
Relates the pulse exit Y(z) to the pulse entry X(z):

$$G(z) = \frac{Y(z)}{X(z)}$$

Pulse transfer function of the system in discrete time

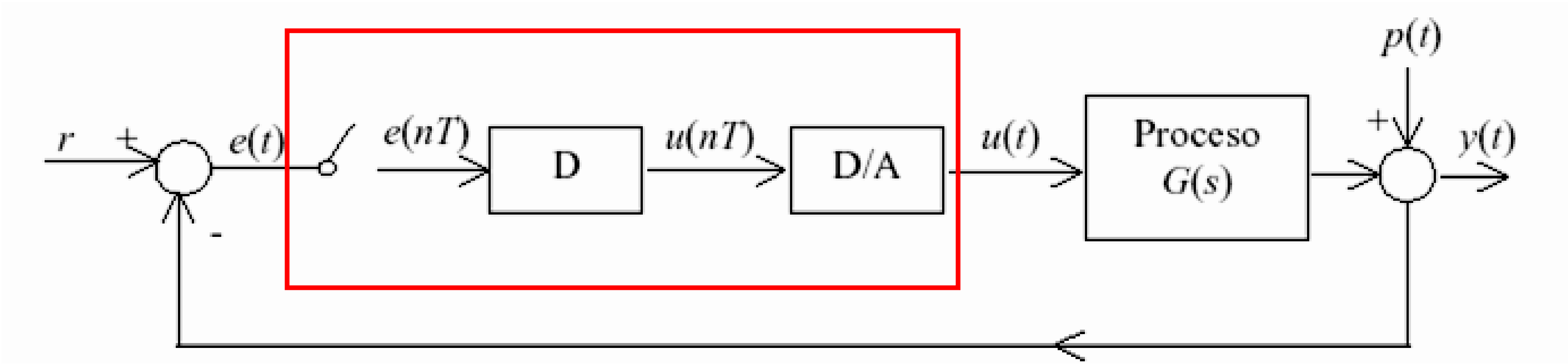
c. Rules to obtain the Z-transfer function

Be careful with the difference between:



Examples

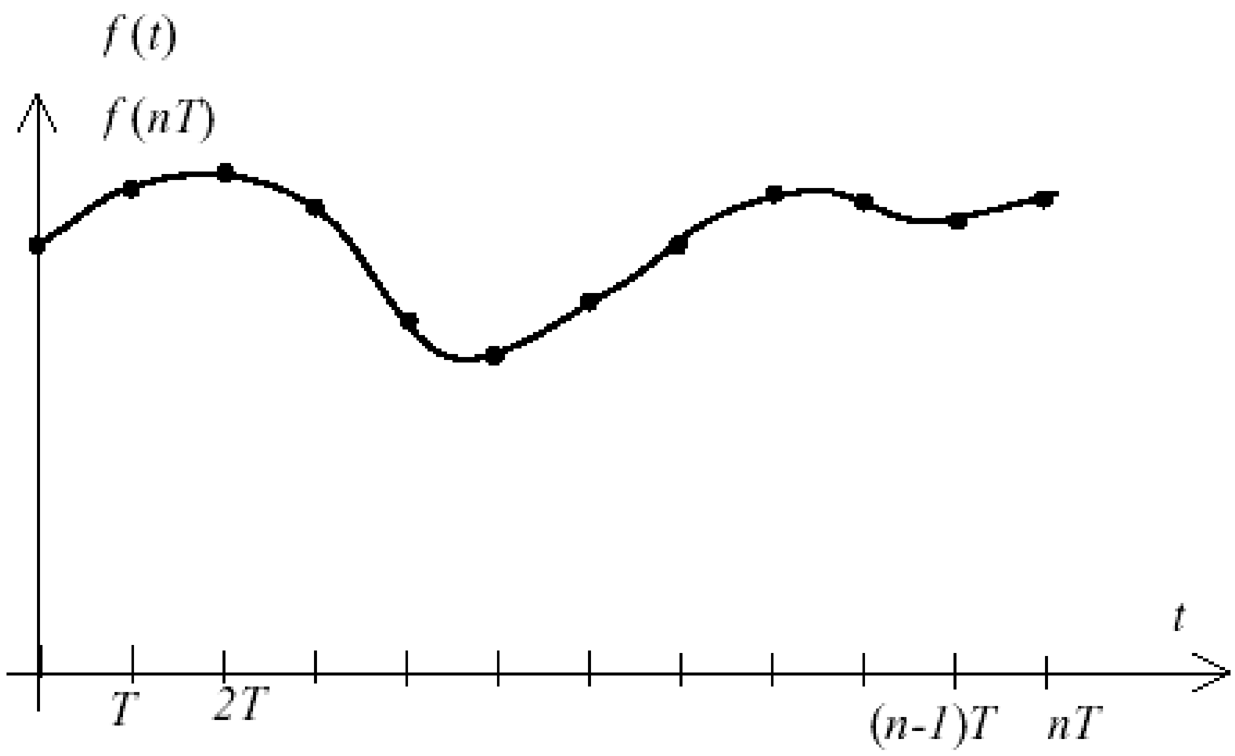
Remember: the idea is to model the following digital control system:



4- Digital control tools

A/D Converter:

It generates an impulse series, each of them weighted by the value of the analogical signal at the corresponding time $t=nT_s$



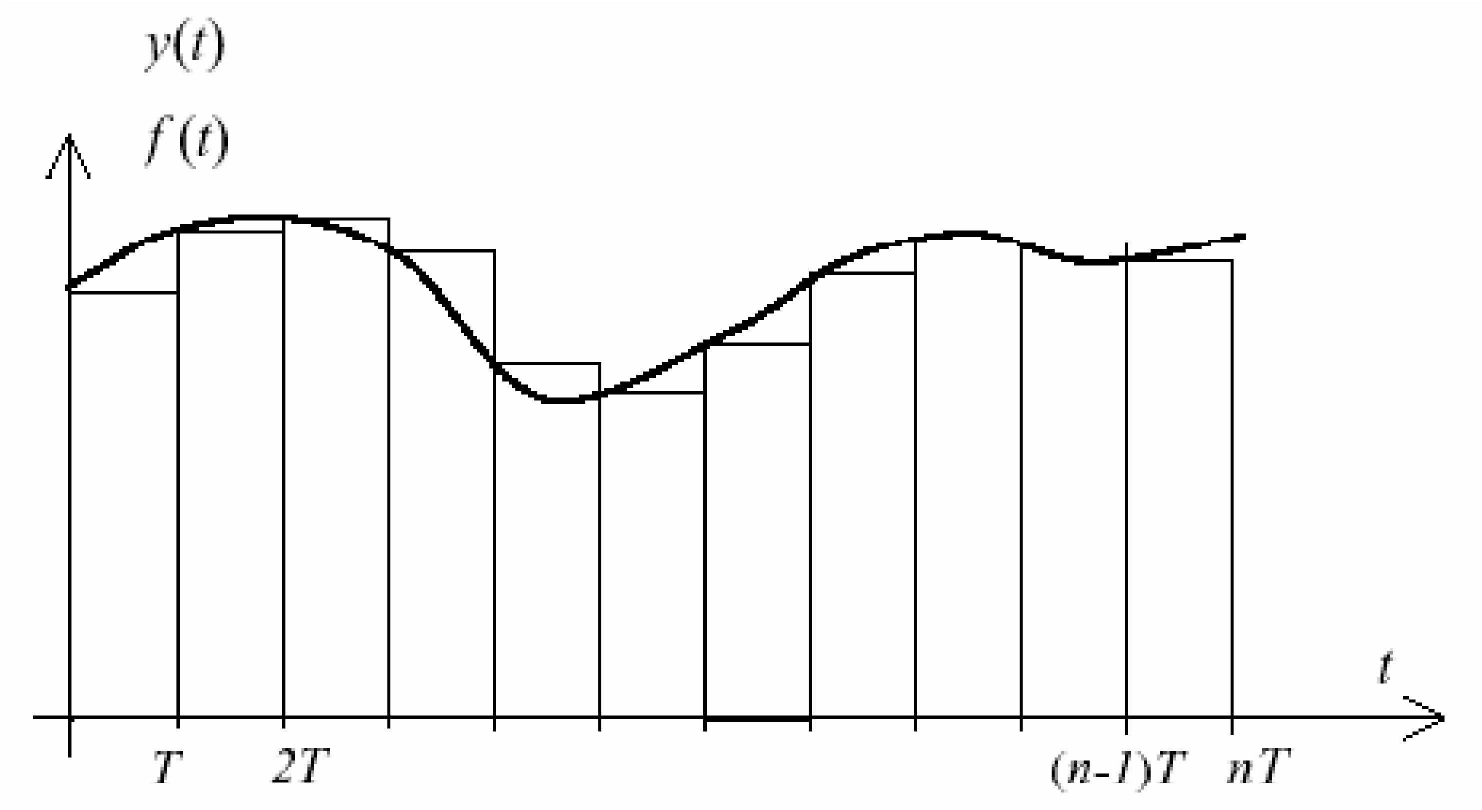
Digital controller:

It processes, through a recursive algorithm, the weights of the entry impulses and it generates (every T_s seconds) an adjusted impulse with the result of the recursive equation.

→ use of a tool to process discrete signals: **Z-transform**

D/A converter: Zero Order Holder (ZOH)

signal re-builder which transforms the impulse series into a stepped signal (analog signal with values for every time instant)

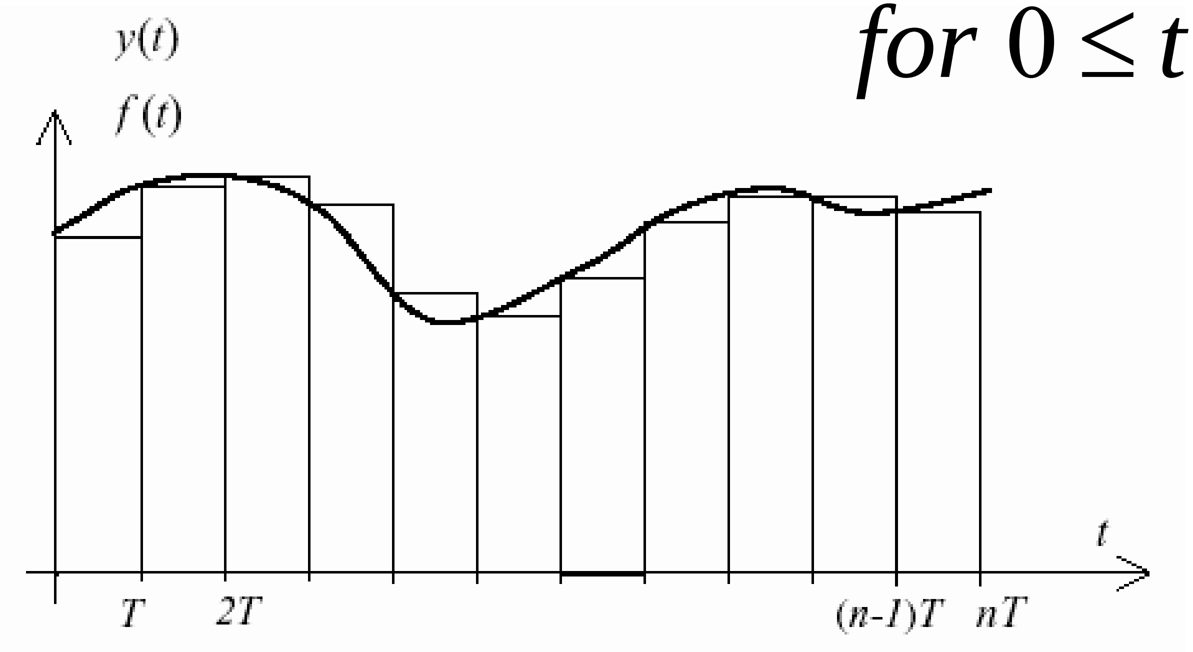


Zero Order Holder (ZOH)

Holder smoothes the sampled signal to produce a constant signal from the last sampled value to the next available sample, i.e.:

$$x_{hold}(kT+t)=x(kT)$$

$$for\ 0\leq t<T$$



Zero Order Holder (ZOH)

Calculation of its transfer function:

Hypothesis: x(t)=0 for t<0

$$x_{hold}(t) = x(0)\big(u(t)-u(t-T)\big)+x(T)\big(u(t-T)-u(t-2T)\big)+\ldots$$
$$+\,x(kT)\big(u(t-kT)-u\big(t-(k+1)T\big)\big)$$

$$\Rightarrow \quad x_{hold}(t)=\sum_{k=0}^{+\infty} x(kT)\big(u(t-kT)-u\big(t-(k+1)T\big)\big)$$

Zero Order Holder (ZOH)

With: $L[u(t)] = \frac{1}{s}$ and $L[u(t-kT)] = \frac{e^{-kTs}}{s}$

$$L[x_{hold}(t)] = \sum_{k=0}^{+\infty} x(kT)\left(\frac{e^{-kTs}}{s} - \frac{e^{-(k+1)Ts}}{s}\right)$$

$$= \frac{1-e^{-Ts}}{s} \sum_{k=0}^{+\infty} x(kT) e^{-kTs}$$

Using: $z=e^{Ts}$, the x(t) z-transform is recognized

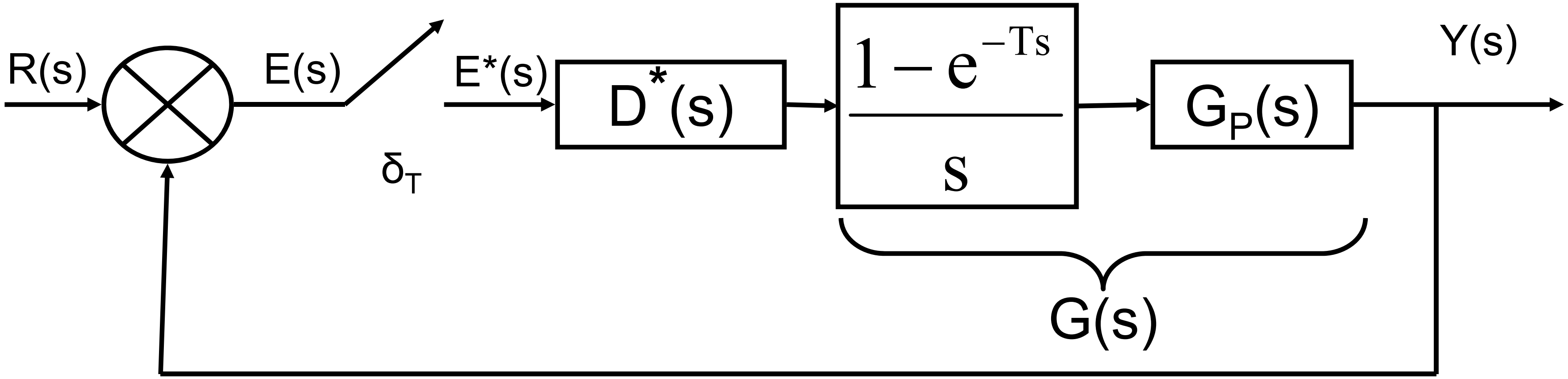
$$X_{hold}(s) = \frac{1-e^{-Ts}}{s} X^*(s)$$

Zero Order Holder (ZOH)

And the transfer function of the zero order holder is obtained:

$$G_{ZOH}(s) = \frac{1 - e^{-Ts}}{s}$$

Pulse transfer function of a digital control system

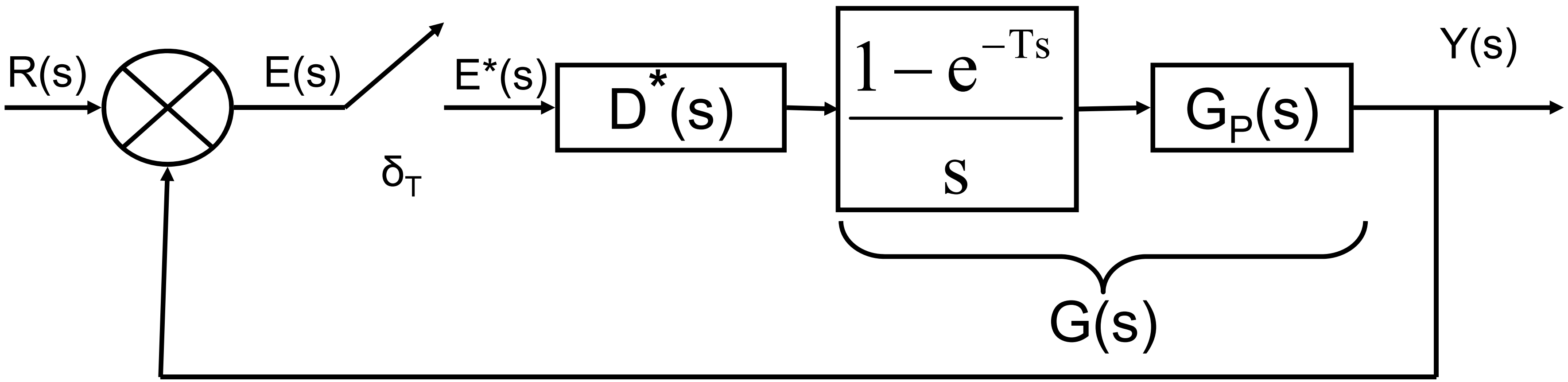


we define:

$$G(s) = \frac{1 - e^{-Ts}}{s} G_P(s)$$

And its z-transform will be computed: G(z).

Pulse transfer function of a digital control system



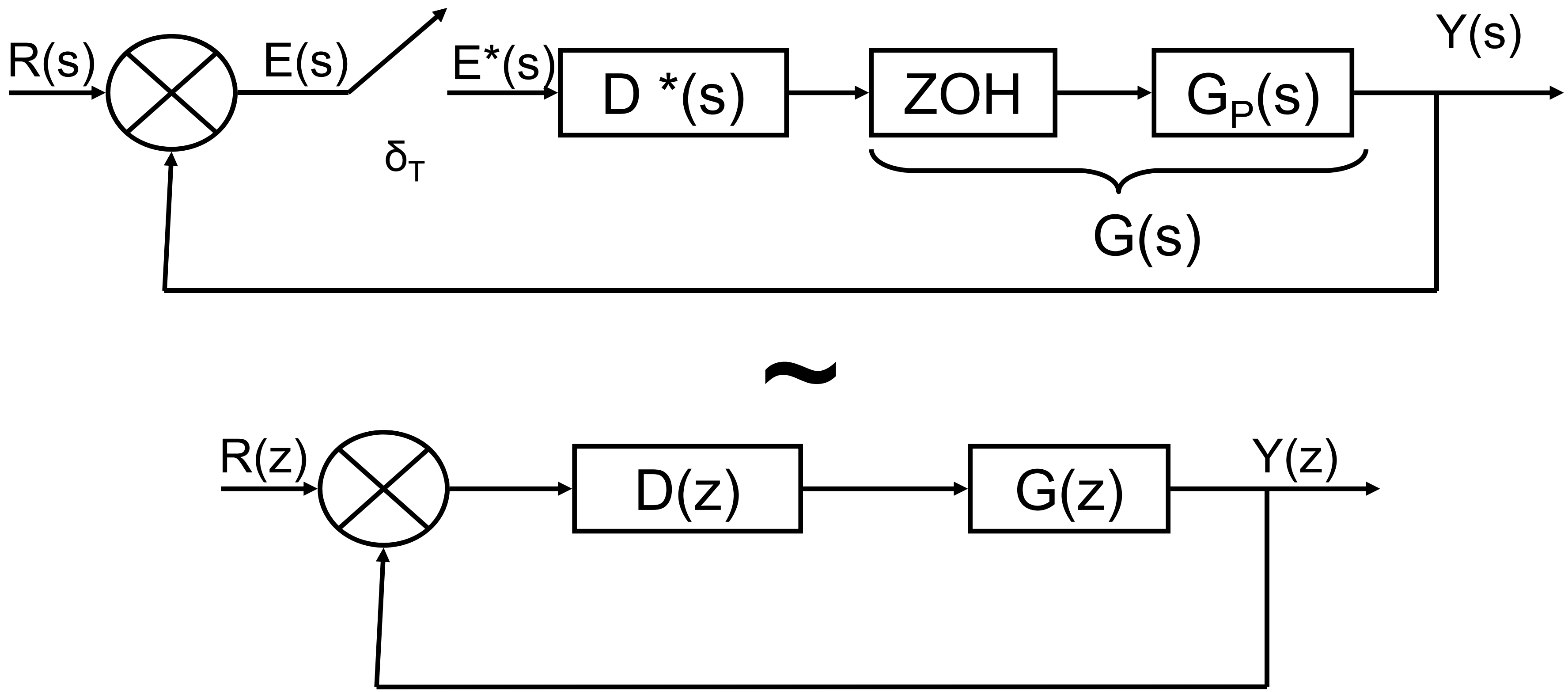
$$\begin{cases} Y(s) = G(s)D^*(s)E^*(s) \\ E(s) = R(s) - Y(s) \end{cases} \Rightarrow \begin{cases} Y^*(s) = G^*(s)D^*(s)E^*(s) \\ E^*(s) = R^*(s) - Y^*(s) \end{cases}$$

$$\Rightarrow Y(z) = G(z)D(z)\left(R(z) - Y(z)\right)$$

$$\frac{Y(z)}{R(z)} = \frac{D(z)G(z)}{1 + D(z)G(z)}$$

Discrete design

Stage 1: compute the transfer function of the continuous part



Discrete design

With: $$G_{ZOH}(s) = \frac{1 - e^{-Ts}}{s}$$

$$Z\left[G_{ZOH}(s) \cdot G_p(s)\right] = Z\left[\frac{1 - e^{-Ts}}{s} G_p(s)\right]$$

$$= Z\left[\left(1 - e^{-Ts}\right)\frac{G_p(s)}{s}\right]$$

$$= Z\left[\frac{G_p(s)}{s}\right] - Z\left[e^{-Ts}\frac{G_p(s)}{s}\right]$$

$$= Z\left[\frac{G_p(s)}{s}\right] - z^{-1} \cdot Z\left[\frac{G_p(s)}{s}\right]$$

Discrete design

The transfer function of the plant + ZOH is deduced:

$$G(z) = \left(1 - z^{-1}\right) Z\left[\frac{G_p(s)}{s}\right]$$

And the transfer function in closed loop of the discrete system:

$$\frac{Y(z)}{R(z)} = \frac{D(z)G(z)}{1 + D(z)G(z)}$$

Discrete design

Stage 2:

To study the characteristics of the closed loop behavior we look for the characteristic equation's roots:

$$1+KG(z)=0 \text{ (for } D(z)=K\text{: proportional controller)}$$

→ **root locus technique**

Construction rules are the same as in the s plane, but the interpretation is different

Example

Relation between the s-plane and the z-plane

When an impulse sampling is incorporated, the complex variables **s** and **z** are related by the equation:

$$z = e^{Ts}$$

→ A pole on the **s** plane can be placed in the **z** plane by this transformation.

Given: $s = \sigma + j\omega$

$$\Rightarrow z = e^{T(\sigma + j\omega)} = e^{T\sigma} e^{jT\omega}$$

$$\boxed{z = e^{T\sigma} e^{j(T\omega + 2k\pi)}}$$

Relation between the s-plane and the z-plane

→ poles and zeros in the **s-**plane, where the frequencies differ in numbers multiples of the sampling frequency $\boldsymbol{\omega_s = 2\pi / T_s}$, belong to the same locations in the **z** plane.

→ relationship between the **z** plane and the **s** plane is not unique

One point in the **z** plane corresponds to an infinite number of points in the **s** plane, but one point in the **s** plane corresponds to only one point in the **z** plane.

Examples

Relation between the s-plane and the z-plane

The root locus is builded the same way as in the continuous domain

but its interpretation differs:

Stability:

s plane: $\sigma<0$

z plane: $|z| = e^{T\sigma} < 1$

Equivalence: **s plane** **z plane**

s plane		z plane
• imaginary axis	~	unitary circle
• left semi-plane	~	inside the circle
• critically stable (s=0)	~	|z|=1 for a pole

→ stability can be determined with the pole positions

→ stability depends on the sampling period T

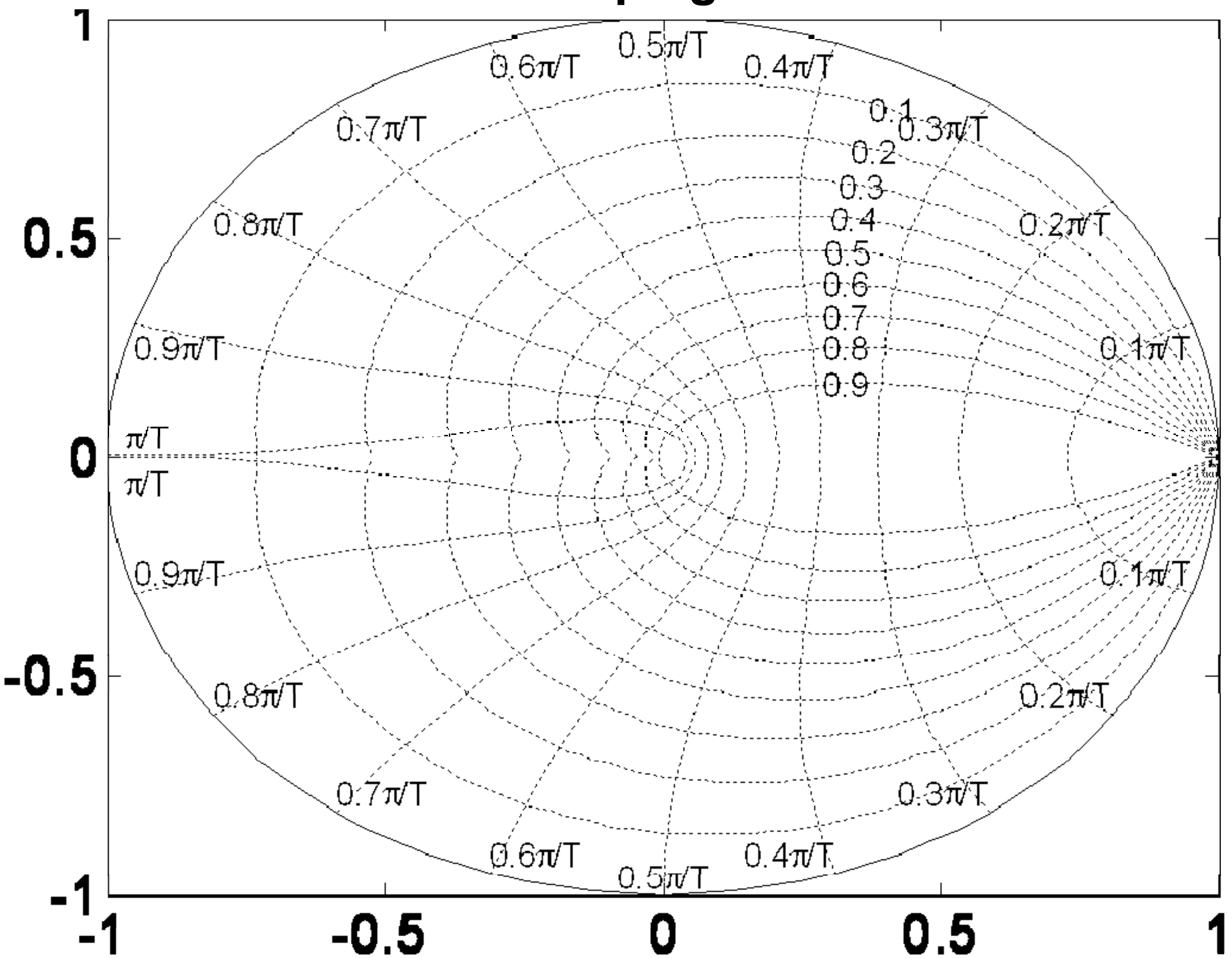
UNIVERSITAT POLITÈCNICA DE CATALUNYA BARCELONATECH

Escola Politècnica Superior de Castelldefels

UNIVERSITAT POLITÈCNICA DE CATALUNYA

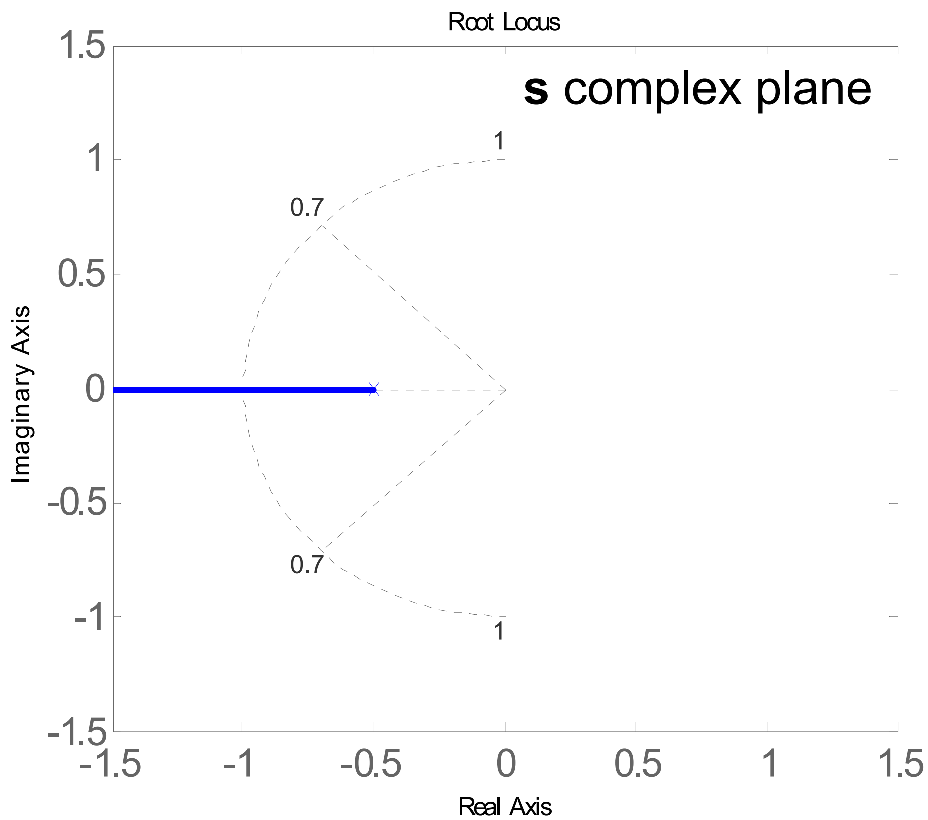
Relation between the s-plane and the z-plane

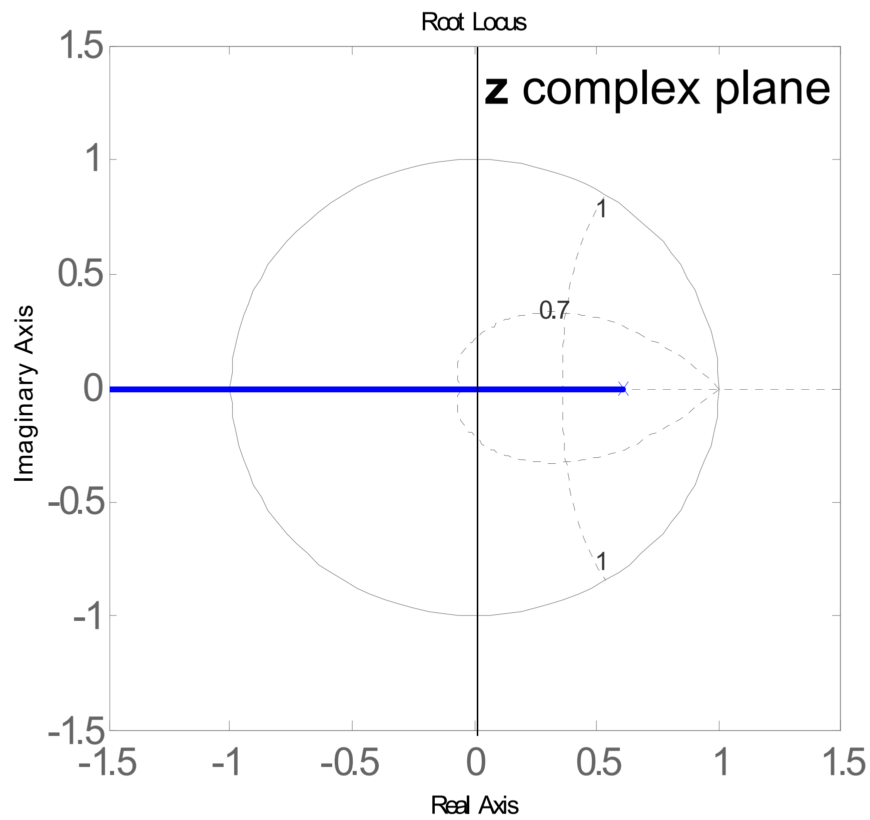
Geometric locus of constant damping factor and natural frequency



Relation between the s-plane and the z-plane

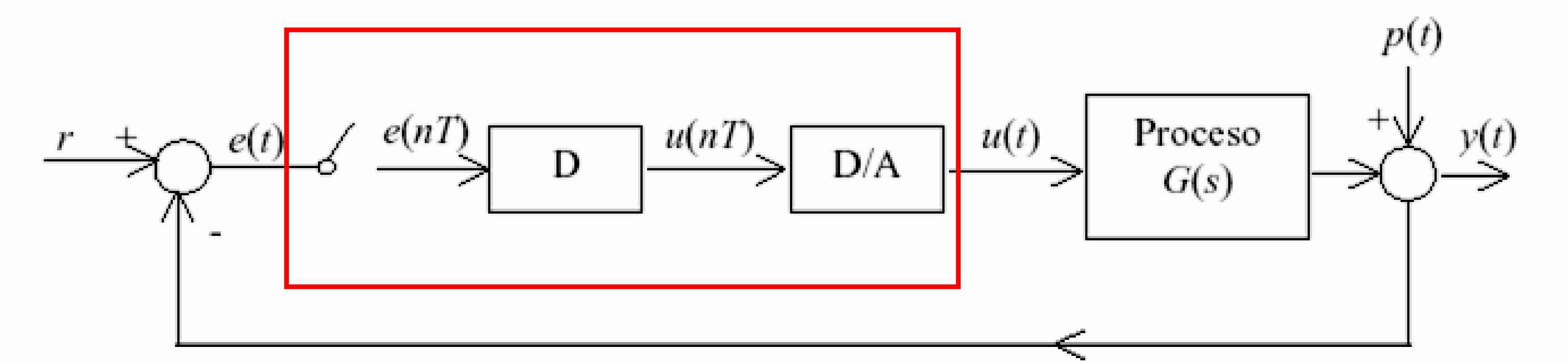
Geometric locus of constant damping factor and natural frequency



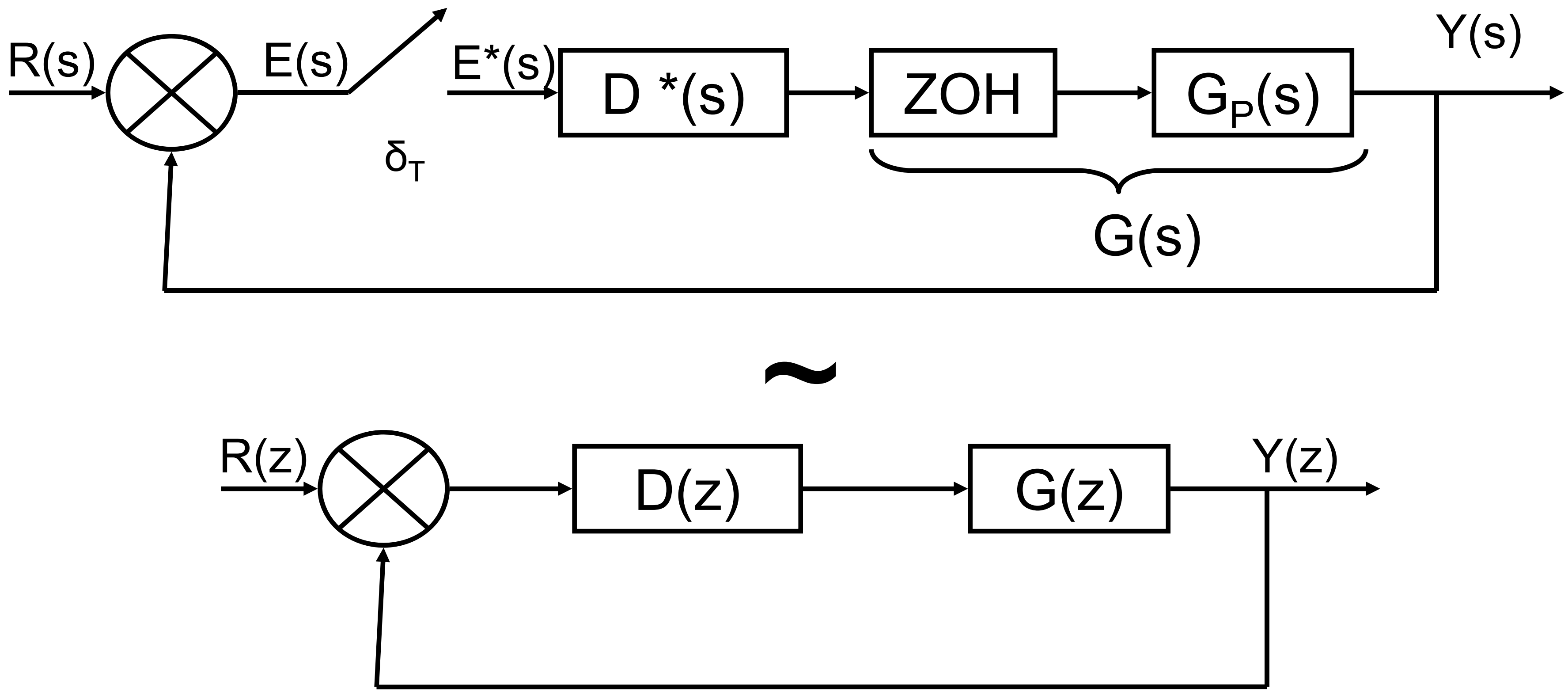


Digital control diagram

Remember: the idea is to model the following digital control system:



Discrete design



Digital controllers

As in continuous systems, both integral or derivative proportional controllers or a combination of them are used to stabilize systems.

Proportional: $u(t)=k_p\, e(t) \rightarrow u(k)=k_p\, e(k)$

$$D(z) = k_p$$

Derivative:

$$u(t) = k_d \frac{de(t)}{dt} \Rightarrow u(k) = k_d\big(e(k) - e(k-1)\big)$$

$$\Rightarrow U(z) = k_d\left(1 - z^{-1}\right)E(z)$$

$$D(z) = k_d\left(1 - z^{-1}\right) = k_d \frac{z-1}{z}$$

Digital controllers

Integrator:

$$u(t) = k_i \int_0^t e(t)dt \Rightarrow u(k) = u(k-1) + k_i e(k)$$

$$\Rightarrow U(z) = z^{-1}U(z) + k_i E(z)$$

$$D(z) = \frac{k_i}{1 - z^{-1}} = \frac{k_i z}{z - 1}$$

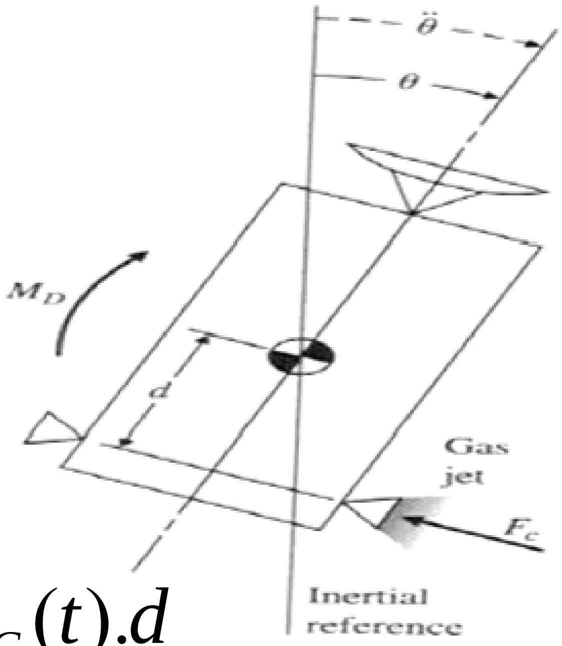
Satellite attitude control system

$$I\ddot{\theta}(t) = M_D + M_I = M_D + F_C(t).d$$

with $\theta(t)$ satellite orientation

M_D torque of the perturbations

$F_C(t)$ applied thrust



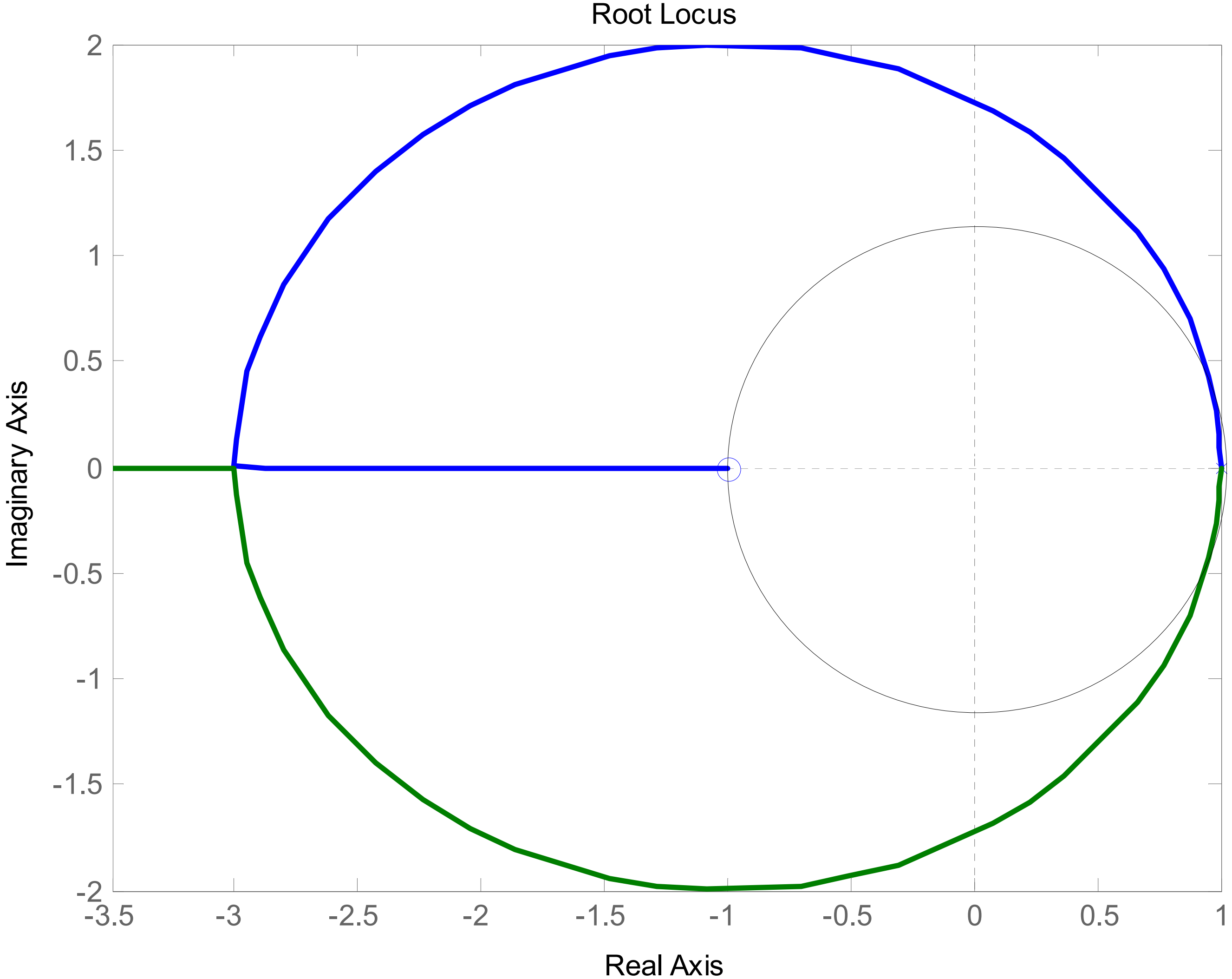
If you consider zero perturbations: $I\ddot{\theta}(t) = F_C(t).d$

or, using the Laplace transform: $G_P(s) = \dfrac{\theta(s)}{d.F_C} = \dfrac{1}{Is^2}$

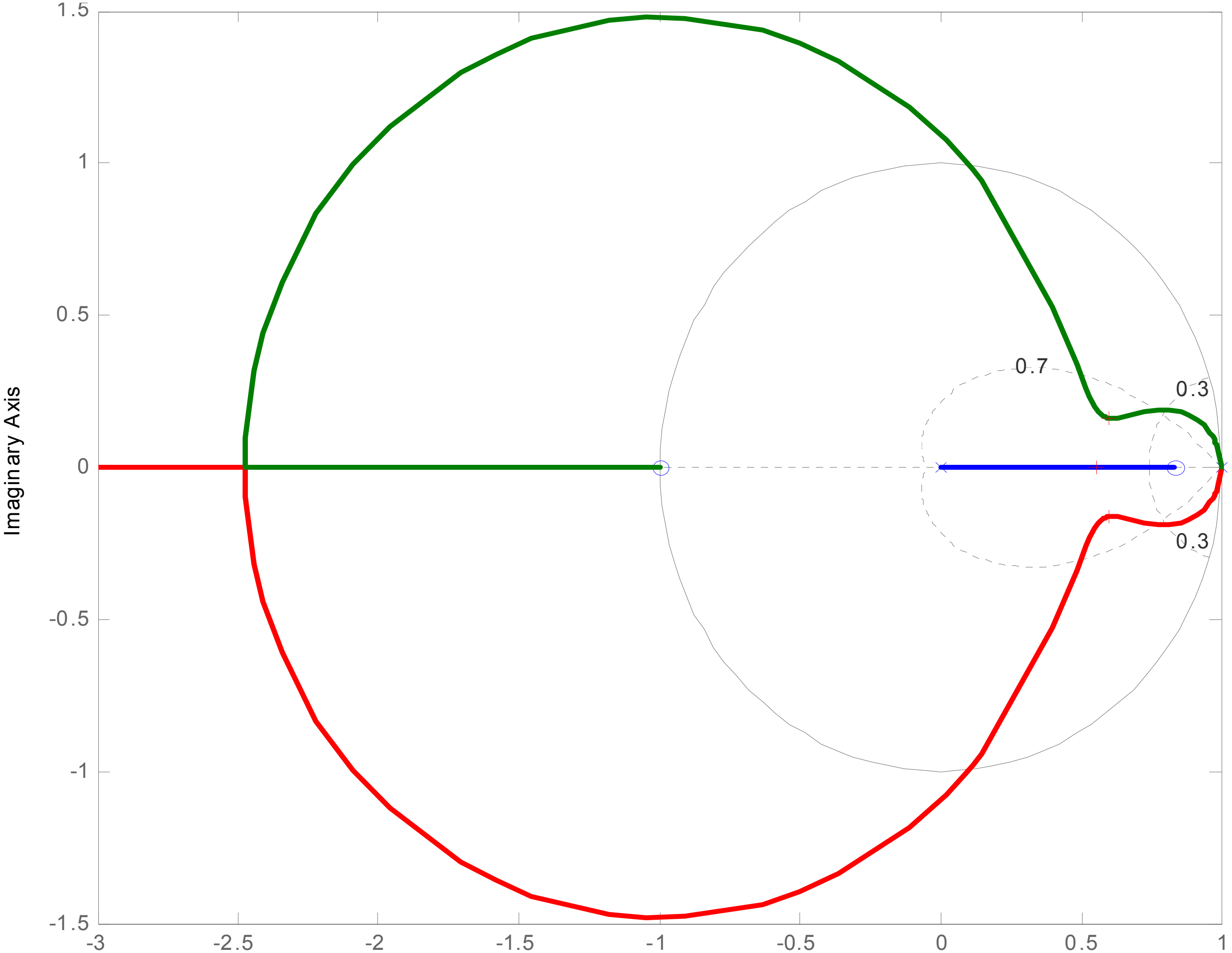
Design requirements: ω_n=0.3rad/s

ζ=0.7

Satellite attitude control system without any controller



Satellite attitude control system with controller



Sampling period influence

Already seen: destabilizing effect of the zero order holder (ZOH).

1. Compute G(z), for a plant: $G_p(s) = \dfrac{1}{s+1}$

   We introduce an integral digital controller: $D(z) = \dfrac{Kz}{z-1}$

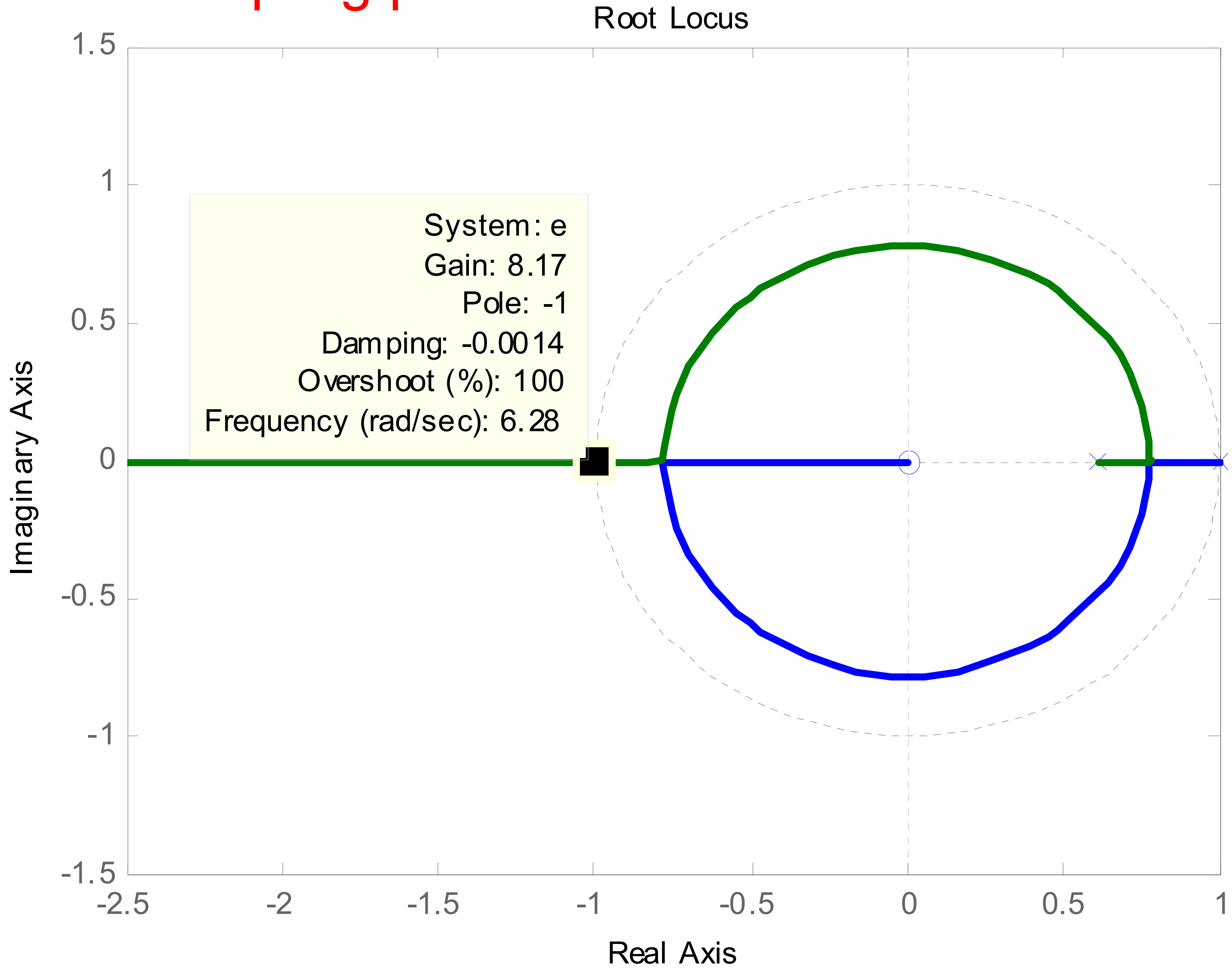
2. Draw the root locus of the transfer function in open loop:

   H(z)=D(z).G(z), for: T=0.5s, 1s y 2sec

3. Compute K_{cr} in the 3 cases

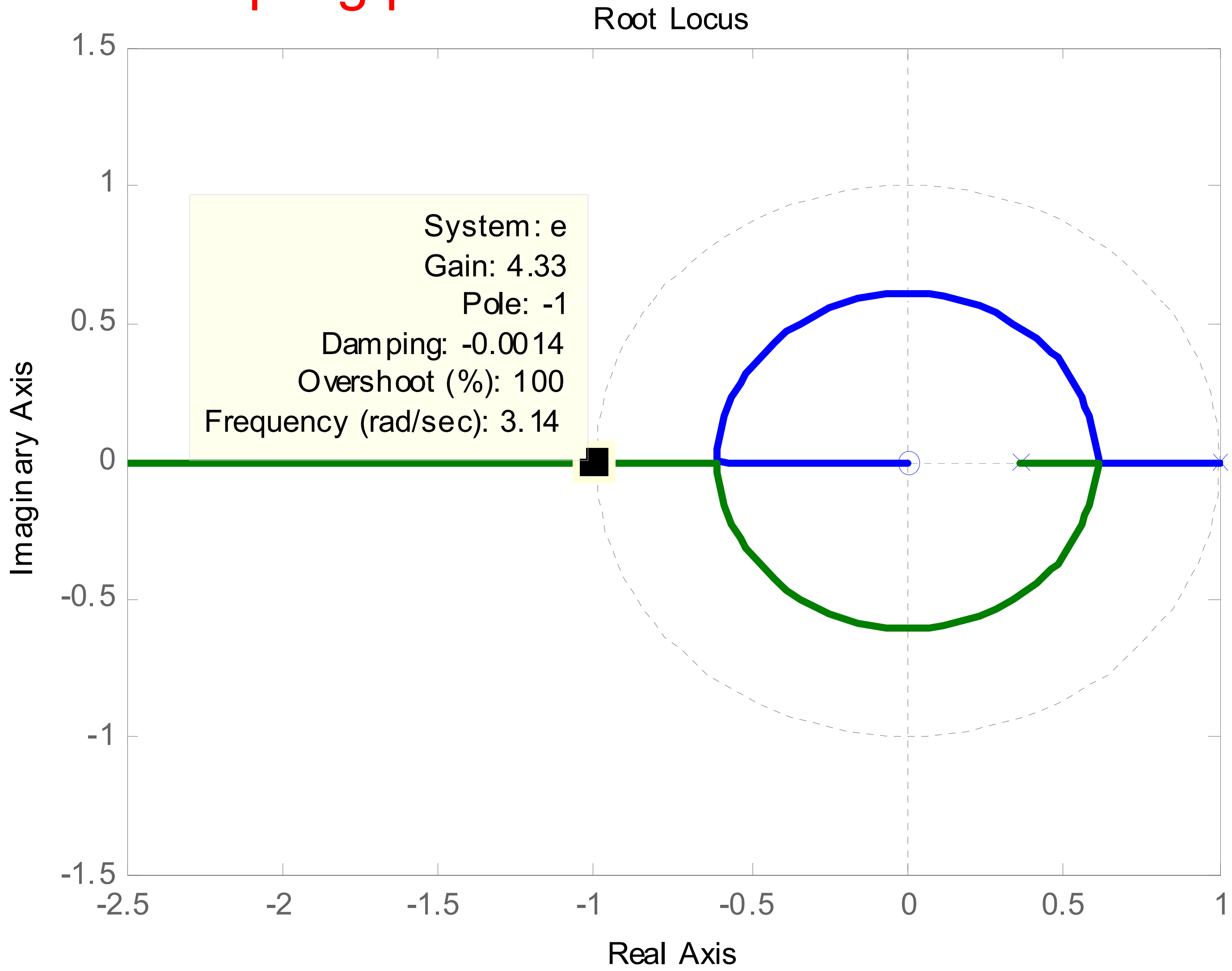
T=0.5sec

Sampling period influence



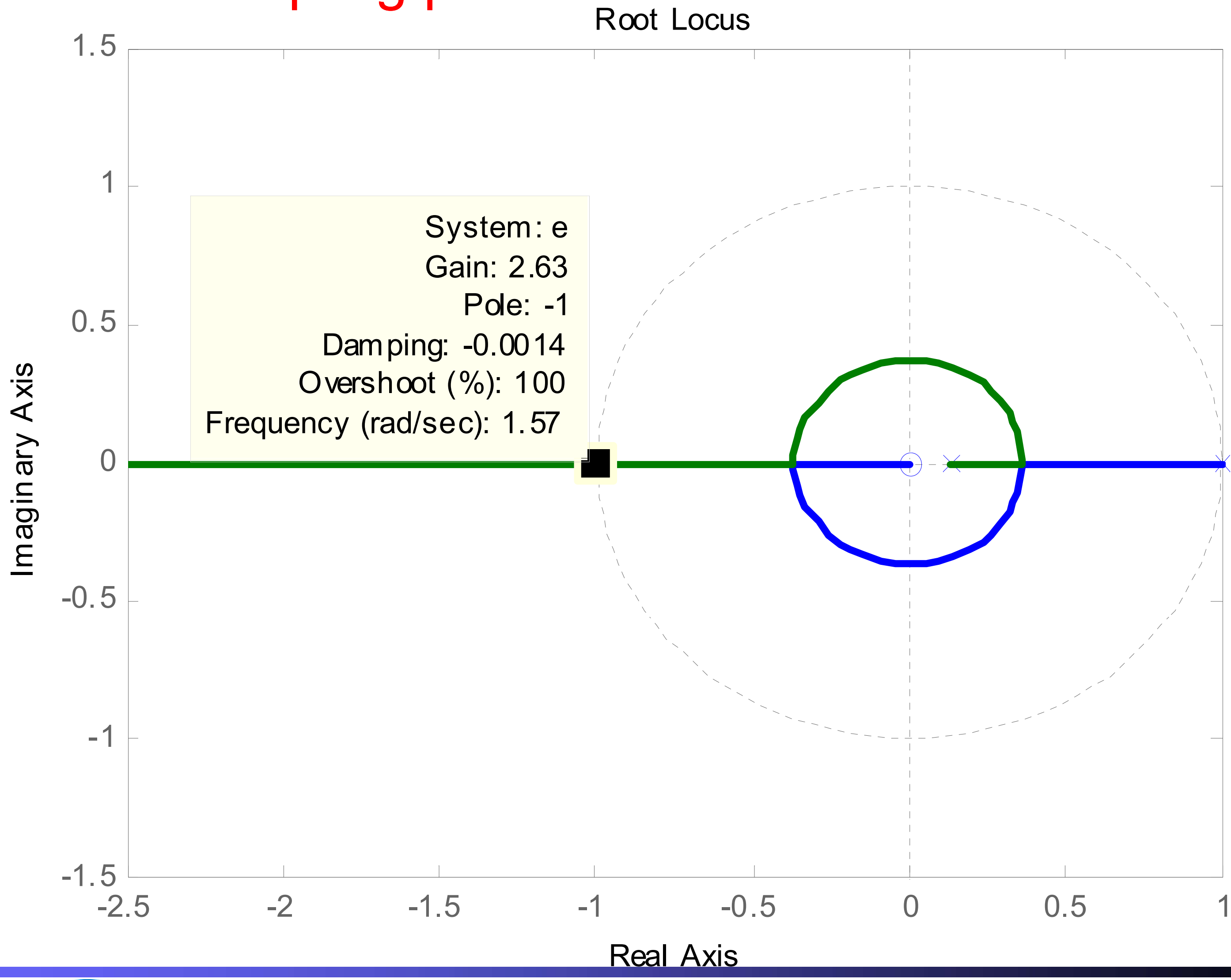
T=1sec

Sampling period influence



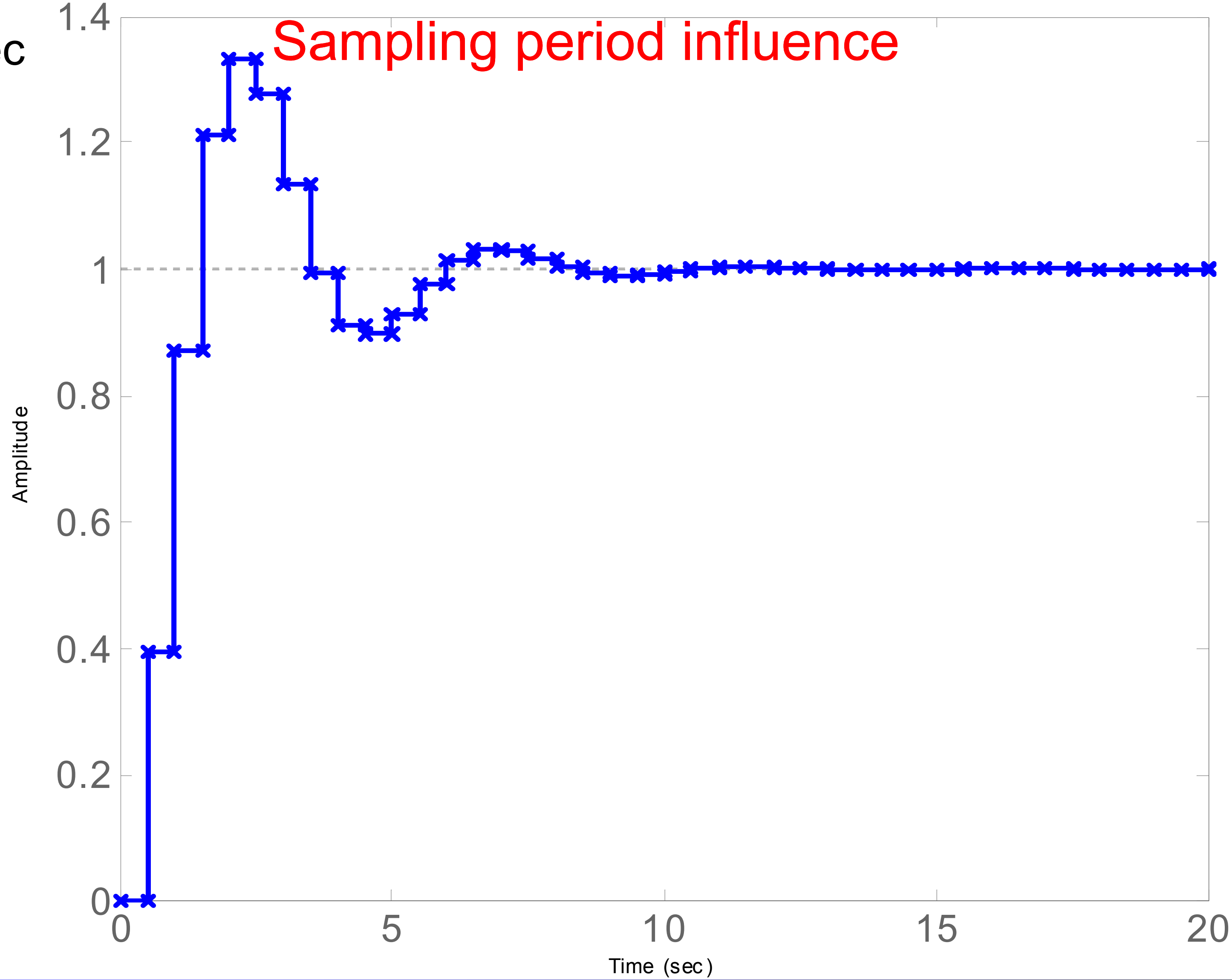
Sampling period influence

T=2sec



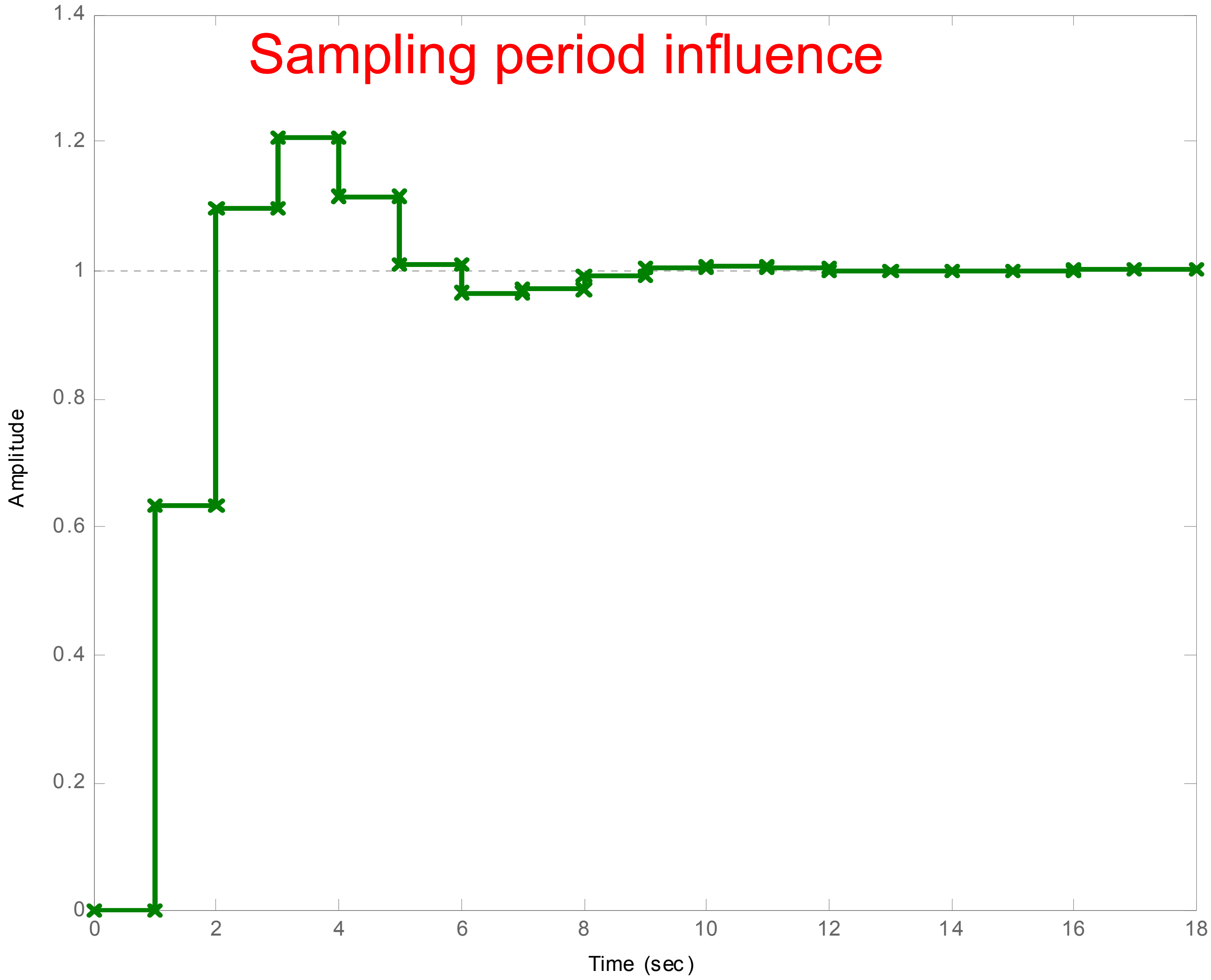
4. Digital control tools

T=0.5sec

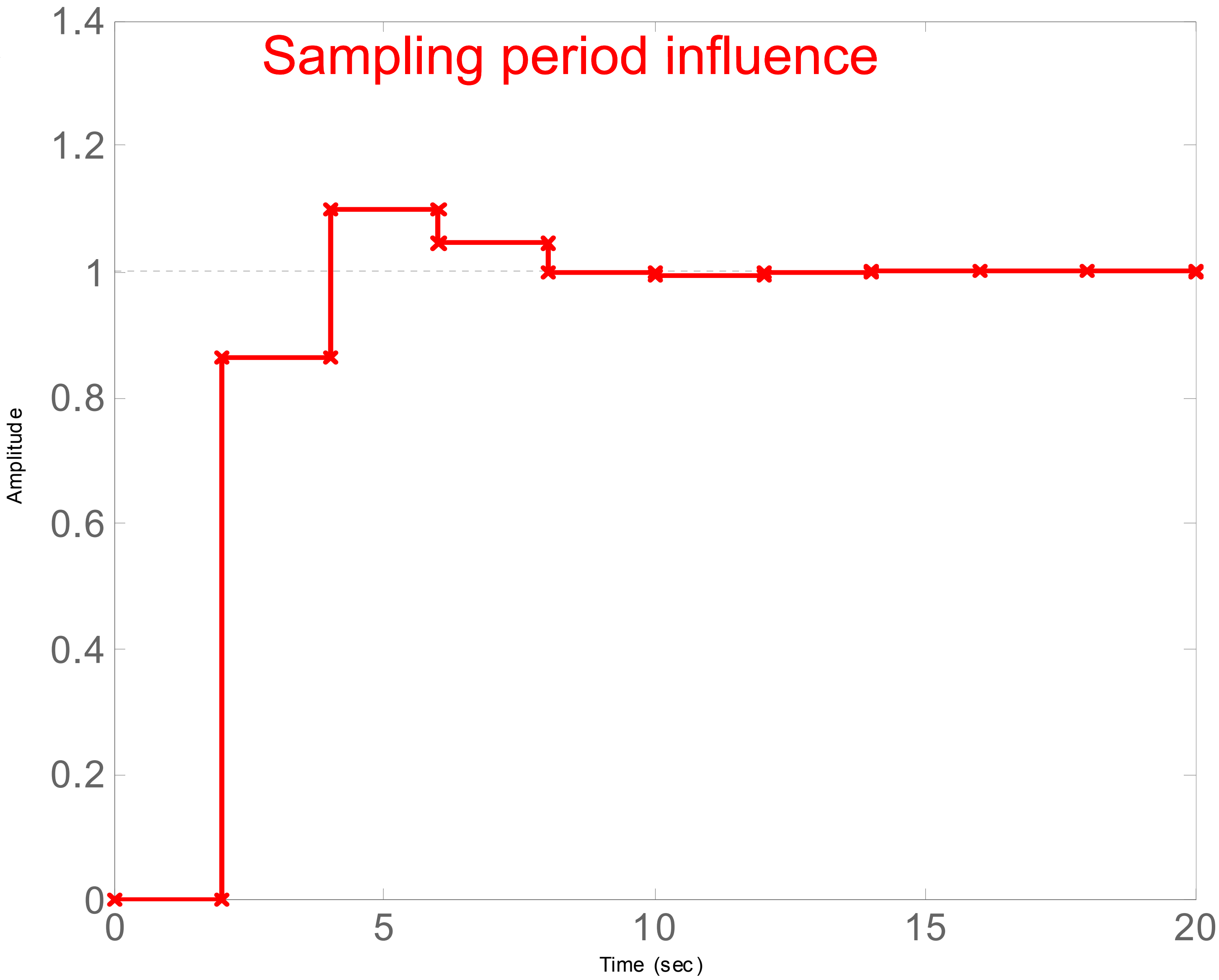


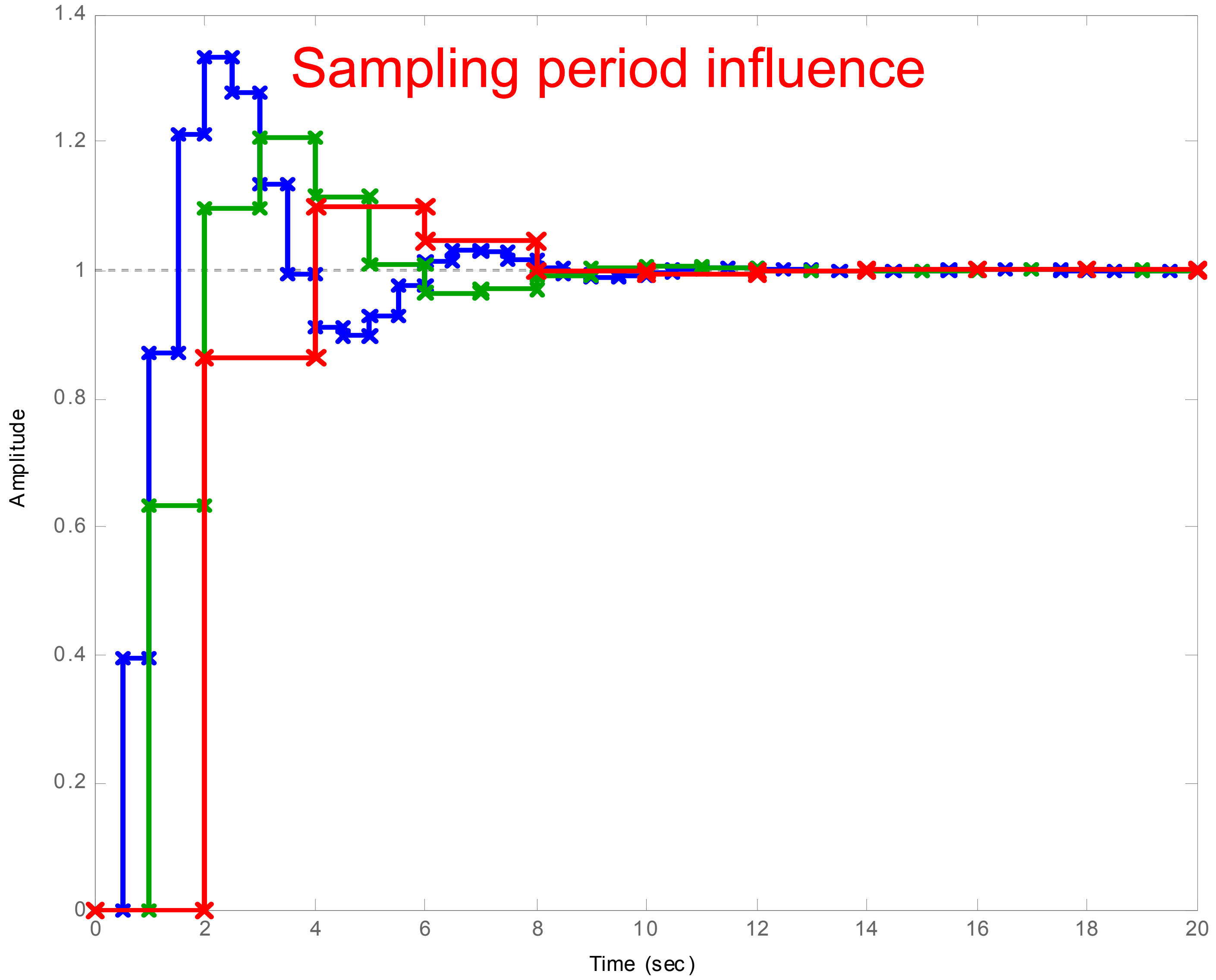
4. Digital control tools

T=1sec



T=2sec


Sampling period influence
1.4
1.2
1
0.8
0.6
0.4
0.2
0
Amplitude
0
5
10
15
20
Time (sec)


Sampling period influence
Amplitude
Time (sec)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
2
4
6
8
10
12
14
16
18
20

Sampling period influence

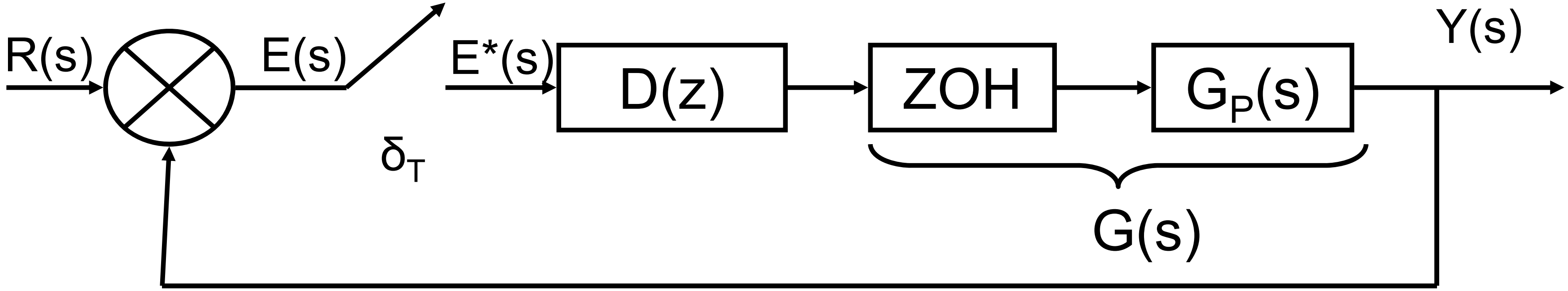
- If the sampling period is small the y(kT) graphic gives a quite precise image of the y(t) response
- Important to select a sampling period based on the satisfaction:
  - of the sampling theorem (Nyquist),
  - of the system dynamics,
  - of the equipment real conditions
- Acceptable rule: 8 to 10 samples per cycle… (for a subdamped system that shows oscillations in the response)

Error in steady state

• Error in steady state: also defined in discrete time.

• Classification depending on the number of open loop poles in the z=1 point (equivalent to s=0: it corresponds to an integrator).

• System's ***type*** defines the characteristics of the system in steady state



Error in steady state

- Error: e(t)=r(t)-y(t)
- **For a stable system** (poles inside the unitary circle):

  Final value theorem gives the error value in steady state at the sampling times:

$$\lim_{k\to\infty} e(kT) = \lim_{z\to 1}\left(\left(1-z^{-1}\right)E(z)\right)$$

$$with\ E(z) = \frac{R(z)}{1+D(z)G(z)}$$

$$and\ G(z) = \left(1-z^{-1}\right)Z\left[\frac{G_p(s)}{s}\right]$$

Error in steady state

$$e_{ss} = \lim_{z \to 1}\left(\left(1 - z^{-1}\right)\frac{R(z)}{1 + D(z)G(z)}\right)$$

Example: Given a digital control system where the plant is a first order system and has a 2 sec. delay and take T=1s

$$G_p(s) = \frac{e^{-2s}}{s+1} \qquad G_{ZOH}(s) = \frac{1 - e^{-Ts}}{s}$$

Compute the error in steady state for a unit step entry.

5. Design method with dead beat response

Method principle:

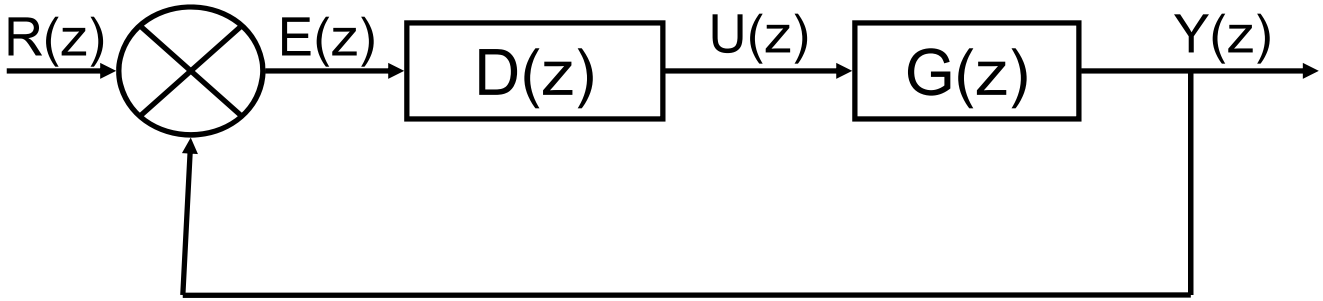
To force the error sequence (for a system subject to a specific entry type, in this course we will always consider a step entry) to reach and keep a zero value after a finite number of sampling periods, in fact, after the minimum possible number of sampling periods

5. Design method with dead beat response

If the response of a closed loop control system to a unitary step entry shows a minimum possible establishment time, with no error in steady state and no oscillatory beat component between sampling instants, then this response type is usually known as **dead beat response**

Drawing

5. Design method with dead beat response



$$\text{with } G(z) = \left(1 - z^{-1}\right) Z\left[\frac{G_p(s)}{s}\right]$$

$$F(z) = \frac{Y(z)}{R(z)} = \frac{D(z)G(z)}{1 + D(z)G(z)}$$

5. Design method with dead beat response

In order to have a finite time of establishment with a zero error in steady state, the system will have to show a finite impulse response:

$$F(z) = \frac{a_0 z^N + a_1 z^{N-1} + \ldots + a_k z^{N-k} + \ldots + a_N}{z^N}$$

or

$$F(z) = a_0 + a_1 z^{-1} + \ldots + a_k z^{-k} + \ldots + a_N z^{-N}$$

$$\text{with } N \geq n \; (n: G_p \text{system order})$$

→ **We are looking for the a_i**

5. Design method with dead beat response

From F(z) the controller transfer function D(z) is calculated

$$D(z) = \frac{F(z)}{G(z)\left(1 - F(z)\right)}$$

5. Design method with dead beat response

Conditions to make the design physically feasible

1. D(z) numerator order ≤ D(z) denominator order (otherwise the controller requires the entry data to be generated after the ones that produce the exit data).

2. If the $G_p(s)$ plant includes an e^{-Ls} transport delay, then the designed closed loop system F(z) has to involve the same delay (otherwise the closed loop system would have to respond before an entry was given).

3. When expanded as a z^{-1} series F(z) and G(z) start with the same term in z^{-1} .

5. Design method with dead beat response

Stability conditions

Avoid the cancellation of an unstable pole of the plant by the use of a digital controller z.

1. If G(z) includes an unstable pole (or critically stable) on z=α

   We define:

$$G(z) = \frac{G_1(z)}{z-\alpha}$$

   and the tf in closed loop:

$$F(z) = \frac{D(z)\dfrac{G_1(z)}{z-\alpha}}{1 + D(z)\dfrac{G_1(z)}{z-\alpha}}$$

5. Design method with dead beat response

Stability conditions

$$1-F(z)=\frac{1}{1+D(z)\dfrac{G_1(z)}{z-\alpha}}=\frac{z-\alpha}{z-\alpha+D(z)G_1(z)}$$

No zero of D(z) cancels G(z)'s pole in z=α if and only if

1-F(z)=0 for z=α

→ **The unstable poles of G(z) must be included as zeros of 1-F(z)**

Stability conditions

2. In the same way for unstable zeros:

$$F(z) = \frac{D(z)G_1(z)(z-\alpha)}{1+D(z)G_1(z)(z-\alpha)}$$

zeros of G(z) that are located on or out the unitary circle must not be cancelled with D(z) poles

→ **The unstable zeros of G(z) must be included as zeros of F(z)**

Design

The error can be written as:

$E(z)=R(z)-Y(z)=R(z)(1-F(z))$

for a step entry:

$$R(z) = \frac{1}{1-z^{-1}}$$

$$\Rightarrow E(z) = \frac{1}{1-z^{-1}}\left(1-F(z)\right)$$

In order to be sure that the system reaches the steady state in a finite number of sampling periods and that maintains a null error output, E(z) must be a *polynomial* in z^{-1} with a finite number of terms

5. Design method with dead beat response

Design

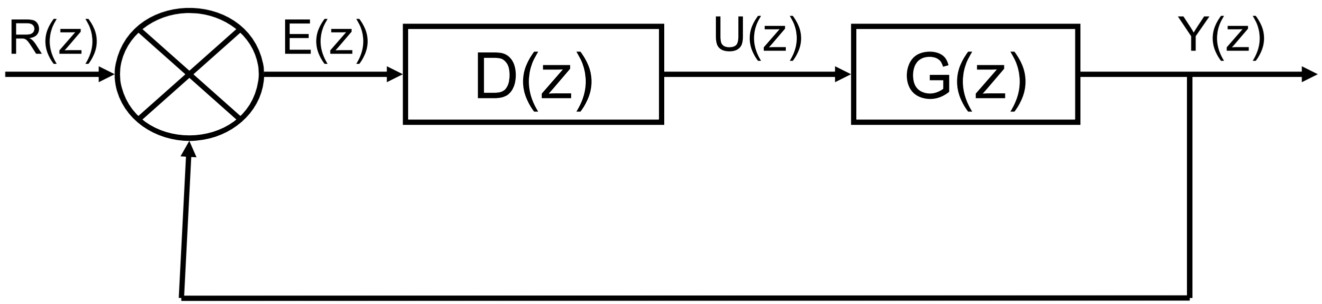
→ 1-F(z) must cancel the denominator: $$E(z) = \frac{1}{1-z^{-1}}\left(1-F(z)\right)$$

$$\Rightarrow 1-F(z) = \left(1-z^{-1}\right)N(z)$$

with **N(z)** polynomial in z^{-1}

Then, E(z)=N(z) is a polynomial in z^{-1} with a **finite number of terms** and e(k) tends to zero in a finite number of sampling periods

Design



For a stable plant $G_p(s)$, the condition so that the exit does not show oscillating components between samplings after the establishment time is:

$$y(t \geq nT) = \textbf{constant} \text{ for a step entry}$$

where **n is the $G_p(s)$ order**

In practice this condition can be applied to u(t)

$$u(t \geq nT) = \textbf{constant} \text{ for a step entry}$$

5. Design method with dead beat response

- Search: $F(z) = \frac{Y(z)}{R(z)} = \frac{D(z)G(z)}{1+D(z)G(z)}$ → $D(z) = \frac{F(z)}{G(z)(1-F(z))}$
- $F(z) = a_0 + a_1 z^{-1} + \ldots + a_k z^{-k} + \ldots + a_N z^{-N}$ with $N \geq n$ ($n: G_p$ order)
- if G_p has a delay, F(z) has the same
- D(z) numerator grade ≤ D(z) denominator grade
- F(z) begins with the same order (in z^{-1}) as G(z)

} Physically feasible

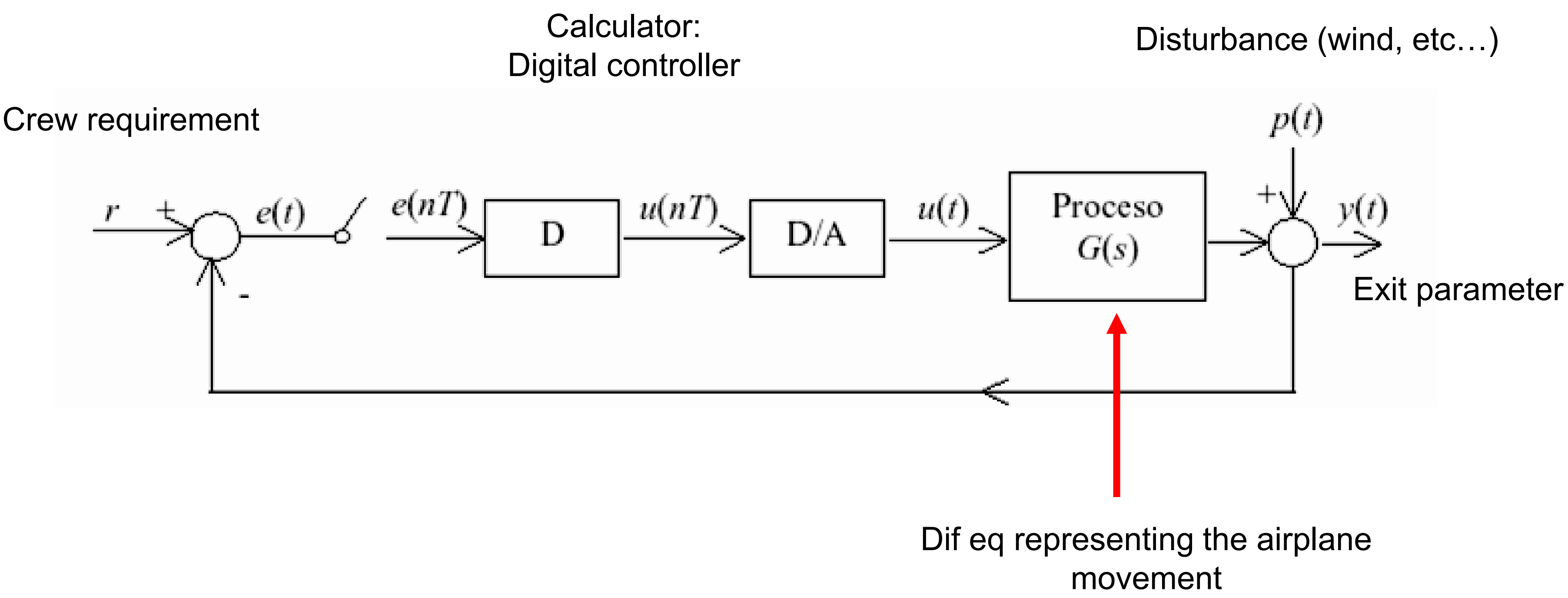
- G(z) unstable poles = (1-F(z)) zeros
- G(z) unstable zeros = F(z) zeros

} Stability condition

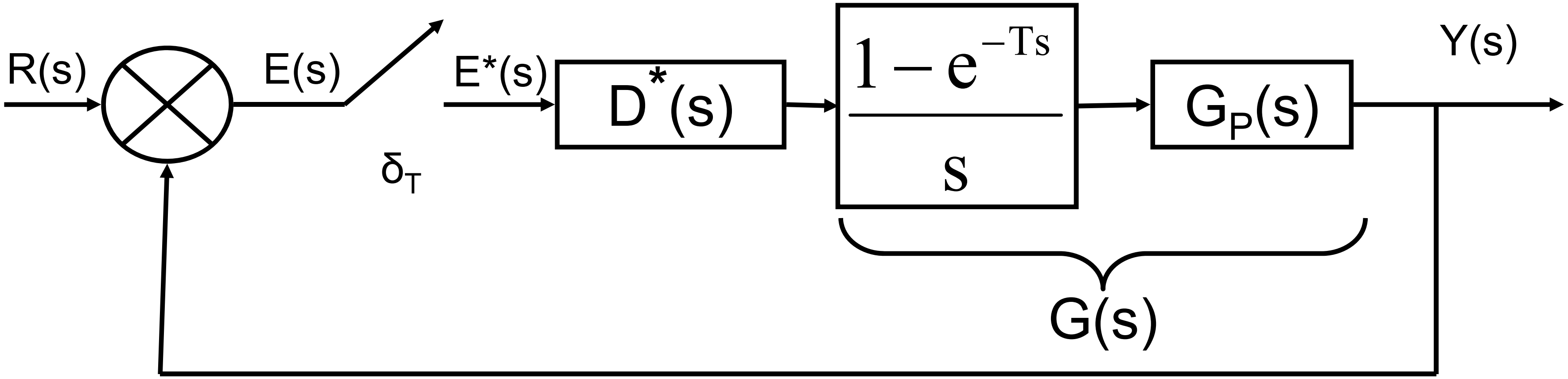
- $1 - F(z) = (1 - z^{-1})N(z)$ with **N(z)** polynomial in z^{-1}, for a step entry
- $y(t \geq nT) = \textbf{constant}$ for a step entry

n is the $G_p(s)$ order

5. Design method with dead beat response



5. Design method with dead beat response

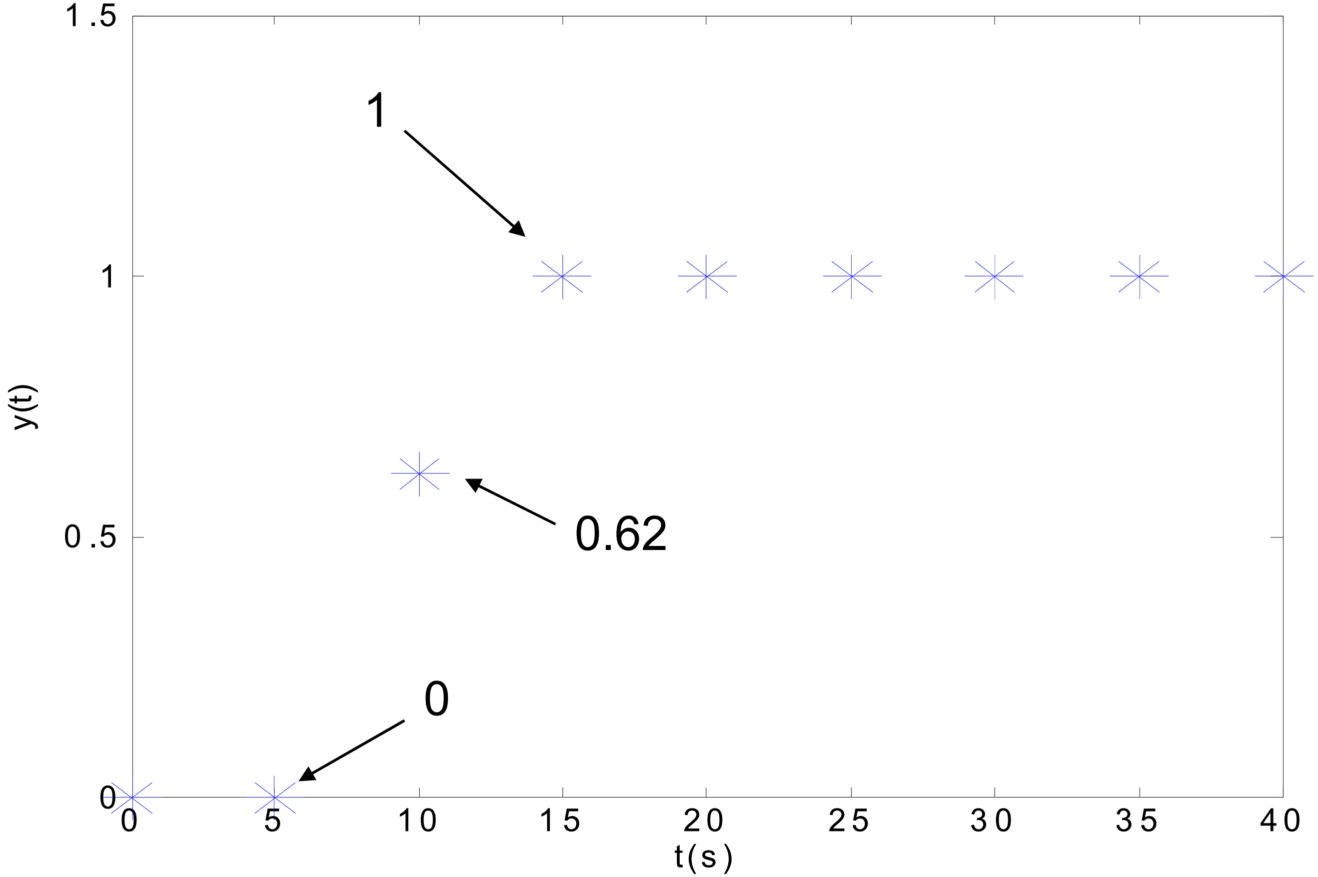


With $G_p(s) = \dfrac{e^{-5s}}{10s+1}$: transfer function of the plant has a 5 sec. delay

T=5s is considered

5. Design method with dead beat response

The following exit y(t) is required for a unitary step entry:



no overshoot nor error in steady state, nor oscillations between samples after reaching a zero error

5. Design method with dead beat response

1. Calculate G(z) (depending on z^{-1})
2. Look for unstable poles and zeros
3. Given the y(nT) sequence, calculate Y(z)
4. Calculate $F(z) = \frac{Y(z)}{R(z)}$
5. Calculate D(z) and verify that it is physically feasible

5. Design method with dead beat response

1. Calculate G(z) (depending on z^{-1})

$$G(z) = \frac{0.39z^{-2}}{1-0.61z^{-1}} = \frac{0.39}{z^2-0.61z}$$

2. Look for unstable poles and zeros

no zero, poles: 0 and 0.61 both stable

5. Design method with dead beat response

3. Given the y(nT) sequence, calculate $$Y(z) = \frac{0.62z^{-2} + 0.38z^{-3}}{1 - z^{-1}}$$

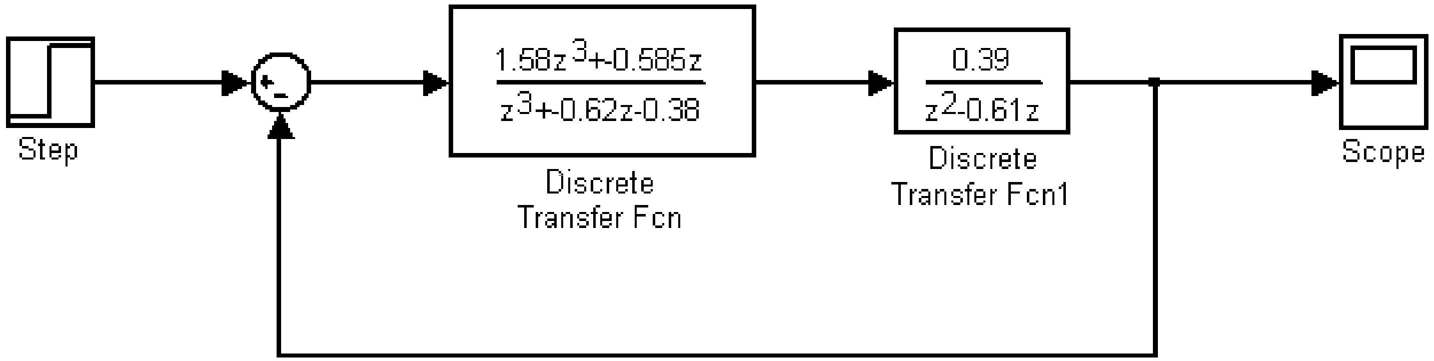
4. Calculate $$F(z) = \frac{Y(z)}{R(z)} = 0.62z^{-2} + 0.38z^{-3}$$

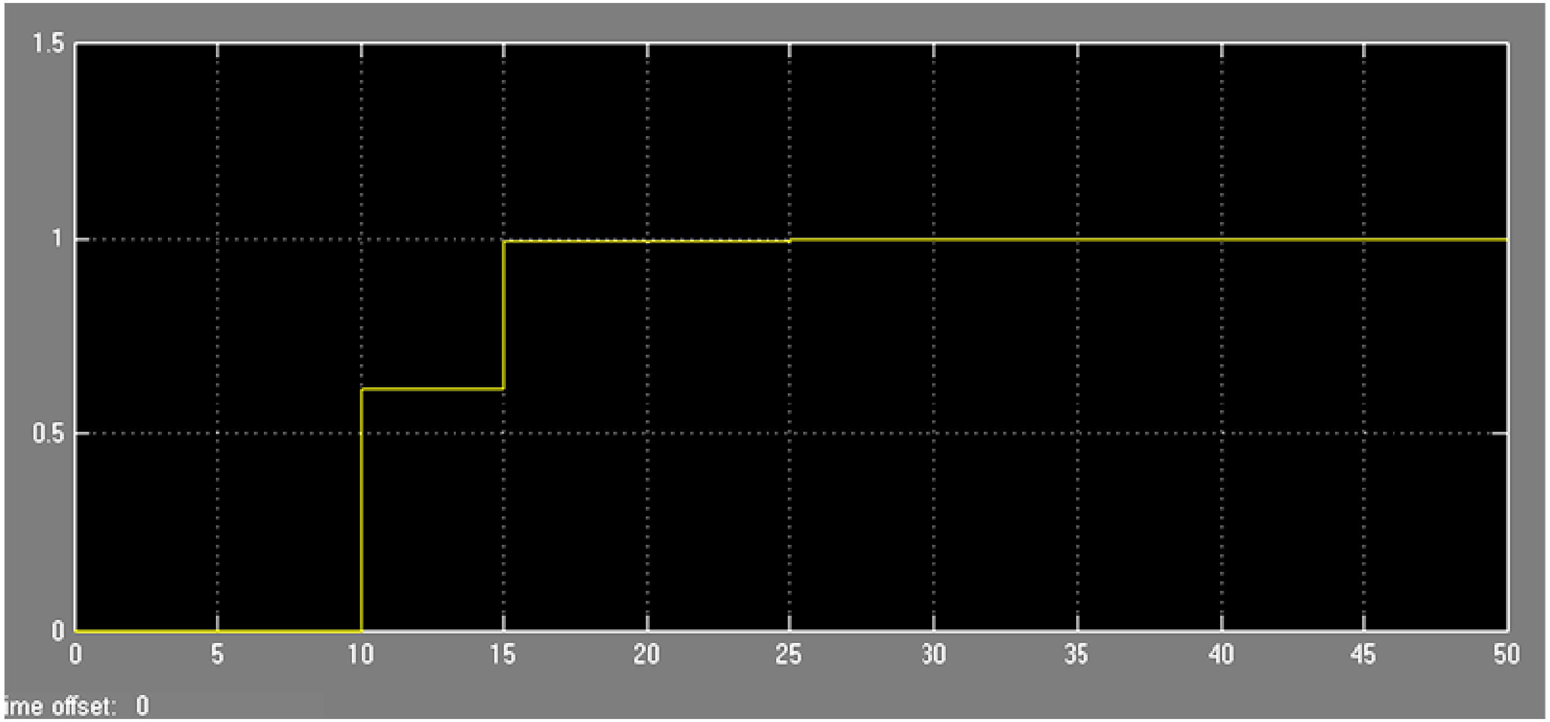
5. Calculate $$D(z) = \frac{1.58(1 - 0.37z^{-2})}{(1 - z^{-1})(1 + z^{-1} + 0.38z^{-2})} = \frac{1.58(z^3 - 0.37z)}{(z-1)(z^2 + z + 0.38)}$$

   and verify that it is physically feasible

$$G(z) = \frac{0.39z^{-2}}{1 - 0.61z^{-1}} = \frac{0.39}{z^2 - 0.61z}$$

5. Design method with dead beat response





REFERENCES

Ogata, K., *Modern Control Engineering* (4th Ed.), Prentice Hall, 2001

Kuo, B., *Automatic Control Systems*, Wiley and Sons, (8th Ed.), 2002

Dorf, R. and Bishop, R., *Modern Control Systems*, (10th Ed.), Prentice Hall, 2004

Kirk, D. E., *Optimal Control Theory: An Introduction*, 2004

Athans, M. A. and Falb, P. L., *Optimal Control*, McGraw-Hill, New York, 1966

Levine, W. S., *The Control Handbook*. New York: CRC Press, 1996